FFT
(Sine Transform)

1D

$$\begin{cases} -u_{xx} = f(x) \\ u = 0 \text{ on } \partial\Omega \\ u(0)=0,\ u(1)=0 \end{cases} \Rightarrow \underbrace{\frac{1}{h^2}\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}}_{\underline{u}} = \underbrace{\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{pmatrix}}_{\underline{F}}$$

Let
$$R = \begin{array}{c} \begin{bmatrix} \sin(\pi x_1) & \sin(2\pi x_1) & \cdots \sin(m\pi x_1) \\ \sin(\pi x_2) & \sin(2\pi x_2) & \sin(m\pi x_2) \\ \vdots & \vdots & \vdots \\ \sin(\pi x_m) & \sin(2\pi x_m) & \sin(m\pi x_m) \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{evector of } A & \text{evector} & \text{evector} \end{array} \end{array}.$$

Note $R^2 = I \cdot \frac{1}{2h}$

$\Rightarrow \boxed{R^{-1} = 2hR}$

$\boxed{h = \frac{1}{m+1}}$

$\boxed{m \text{ interior pts}}$

Evector Decomposition of A:

$$\lambda_j = \frac{2}{h^2}\left(\cos(j\pi h) - 1\right)$$

$$AR = R\Lambda \Rightarrow A = R\Lambda R^{-1}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{j\ldots} \end{pmatrix}$$

With the decomposition:

$$AU = F$$

$$R\Lambda R^{-1} u = F$$

$$u = \underbrace{R\Lambda^{-1}\underbrace{R^{-1}F}_{\hat{F}}}_{\hat{u}}$$

$\Rightarrow$ Define
$$\hat{F} = R^{-1}F$$
$$u = R\hat{u} \Rightarrow \hat{u} = R^{-1}u$$

Steps:

(idst) 1) Compute $\hat{F} = R^{-1}F = 2hRF$ (transform F)

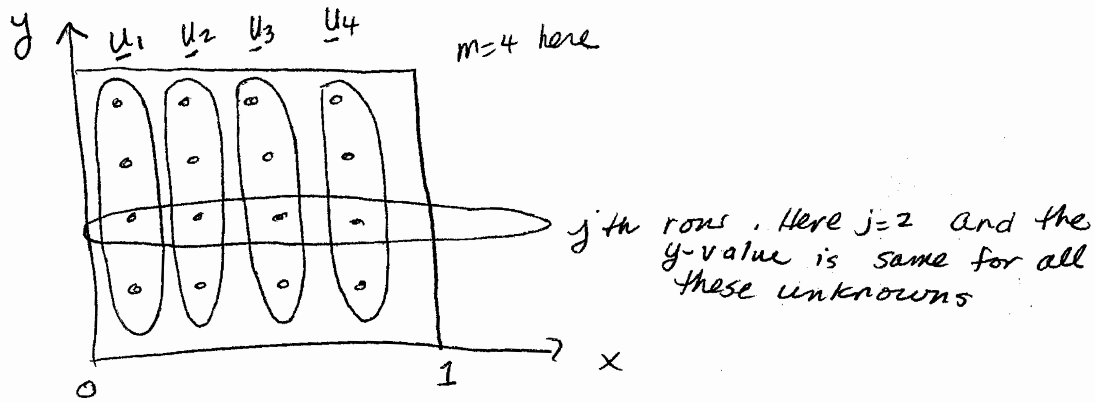
2) Compute $\hat{u} = \Lambda^{-1}\hat{F}$ (diagonal multiply)

(dst) 3) Compute $u = R\hat{u}$ (inverse transform $\hat{u}$)

2D Let's see how to use the 1D transform for $-u_{xx}$ to solve the 2D Poisson problem

$$\begin{cases} -u_{xx} - u_{yy} = f(x,y) \\ u = 0 \text{ on } \partial\Omega \\ \Omega : \text{unit square} \end{cases}$$

Think of organizing groups of unknowns in the vertical direction (y-direction) as shown below.



Discretize

$$A \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_m \end{pmatrix} = \begin{pmatrix} \underline{F}_1 \\ \underline{F}_2 \\ \vdots \\ \underline{F}_m \end{pmatrix}.$$

Write A into its horizontal and vertical stencil weights based on this ordering of the unknowns. think

$$A = \begin{matrix} 1/h^2 \\ -2/h^2 \\ 1/h^2 \end{matrix} \;+\; \frac{1}{h^2} \quad -2/h^2 \quad \frac{1}{h^2}$$

$$A\underline{u} = \frac{1}{h^2}\begin{pmatrix} -2I & I & & \\ I & -2I & I & \\ & \ddots & \ddots & I \\ & & I & -2I \end{pmatrix}\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_m \end{pmatrix} + \begin{pmatrix} T & & & \\ & T & & \\ & & \ddots & \\ & & & T \end{pmatrix}\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_m \end{pmatrix}$$

↗ horizontal ↑ vertical

$$T = \begin{pmatrix} -2/h^2 & +\frac{1}{h^2} & & \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \\ & \ddots & \ddots & \frac{1}{h^2} \\ & & \frac{1}{h^2} & -\frac{2}{h^2} \end{pmatrix}$$

We know $TR = R\Lambda$ & $R^{-1}TR = \Lambda$.

So multiply $A\underline{u} = \underline{F}$ on left and right sides by

$$\begin{pmatrix} R^{-1} & & \\ & R^{-1} & \\ & & \ddots \\ & & & R^{-1} \end{pmatrix}$$ to get

$$\begin{pmatrix} R^{-1} & & \\ & \ddots & \\ & & R^{-1} \end{pmatrix} \begin{pmatrix} -2I & I & & \\ I & -2I & I & \\ & \ddots & \ddots & I \\ & & I & -2I \end{pmatrix} \cdot \frac{1}{h^2} \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_m \end{pmatrix} + \underbrace{\begin{pmatrix} R^{-1} & & \\ & \ddots & \\ & & R^{-1} \end{pmatrix} \begin{pmatrix} T & & \\ & T & \\ & & \ddots & \\ & & & T \end{pmatrix} \begin{pmatrix} R & & \\ & R & \\ & & \ddots & \\ & & & R \end{pmatrix}}_{\begin{pmatrix} \Lambda & & & \\ & \Lambda & & \\ & & \ddots & \\ & & & \Lambda \end{pmatrix}} \left\{ \begin{pmatrix} \hat{\underline{u}}_1 \\ \hat{\underline{u}}_2 \\ \hat{\underline{u}}_m \end{pmatrix} \right\}$$

$$= \begin{pmatrix} R^{-1} & & \\ & R^{-1} & \\ & & \ddots \\ & & & R^{-1} \end{pmatrix} \begin{pmatrix} \underline{F}_1 \\ \underline{F}_2 \\ \underline{F}_m \end{pmatrix}$$

$$\frac{1}{h^2} \begin{pmatrix} -2R^{-1} & R^{-1} & & \\ R^{-1} & -2R^{-1} & R^{-1} & \\ & \ddots & \ddots & R^{-1} \\ & & R^{-1} & -2R^{-1} \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_m \end{pmatrix} + \begin{pmatrix} \Lambda \hat{\underline{u}}_1 \\ \Lambda \hat{\underline{u}}_2 \\ \vdots \\ \Lambda \hat{\underline{u}}_m \end{pmatrix} = \begin{pmatrix} \hat{\underline{F}}_1 \\ \hat{\underline{F}}_2 \\ \vdots \\ \hat{\underline{F}}_m \end{pmatrix}$$

Remember: $R^{-1}\underline{u}_j = \hat{\underline{u}}_j$

$$\boxed{\frac{1}{h^2} \begin{pmatrix} -2\hat{\underline{u}}_1 + \hat{\underline{u}}_2 \\ \hat{\underline{u}}_1 - 2\hat{\underline{u}}_2 + \hat{\underline{u}}_3 \\ \vdots \\ \hat{\underline{u}}_{m-1} - 2\hat{\underline{u}}_m \end{pmatrix} + \begin{pmatrix} \Lambda \hat{\underline{u}}_1 \\ \Lambda \hat{\underline{u}}_2 \\ \vdots \\ \Lambda \hat{\underline{u}}_m \end{pmatrix} = \begin{pmatrix} \hat{\underline{F}}_1 \\ \hat{\underline{F}}_2 \\ \vdots \\ \hat{\underline{F}}_m \end{pmatrix}.}$$

Look carefully at the jth row of each of the m blocked equations above to see that we get a tridiagonal system to solve for the $\hat{u}$-values in the jth row as shown in the figure (y-fixed for these values).

We get

$$\frac{1}{h^2}(-2\hat{u}_{1j} + \hat{u}_{2j}) + \lambda_j \hat{u}_{1j} = \hat{F}_{1j}$$

$$\vdots$$

$$\frac{1}{h^2}(\hat{u}_{i-1,j} - 2\hat{u}_{ij} + \hat{u}_{i+1,j}) + \lambda_j \hat{u}_{ij} = \hat{F}_{ij}$$

$$\vdots$$

$$\frac{1}{h^2}(\hat{u}_{m-1,j} - 2\hat{u}_{mj}) + \lambda_j \hat{u}_{mj} = \hat{F}_{mj}$$

$$\Downarrow$$

$$\begin{pmatrix} \lambda_j - \frac{2}{h^2} & \frac{1}{h^2} & & 0 \\ \frac{1}{h^2} & \lambda_j - \frac{2}{h^2} & \ddots & \\ & \ddots & \ddots & \frac{1}{h^2} \\ 0 & & \frac{1}{h^2} & \lambda_j - \frac{2}{h^2} \end{pmatrix} \begin{pmatrix} \hat{u}_{1j} \\ \hat{u}_{2j} \\ \vdots \\ \hat{u}_{mj} \end{pmatrix} = \begin{pmatrix} \hat{F}_{1j} \\ \hat{F}_{2j} \\ \vdots \\ \hat{F}_{mj} \end{pmatrix}$$

a tridiagonal system to solve for the $\hat{u}$ values in the jth row (y_j fixed).

Let j vary from 1 to m and solve m independent linear tridiagonal systems.

Once, all m tridiagonals solved, we know $\hat{u}$ all all m^2 interior points. Now, think ~~that~~ $\hat{u}$ is again organized by columns, $\underline{\hat{u}}_1, \ldots, \underline{\hat{u}}_m$

and recover $\underline{u}$,

$$\begin{pmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_m \end{pmatrix} = \underline{u} = \begin{pmatrix} R\,\underline{\hat{u}}_1 \\ R\,\hat{u}_2 \\ \vdots \\ R\,\hat{u}_m \end{pmatrix}.$$