Traffic Flow

First approach — track motion of each car

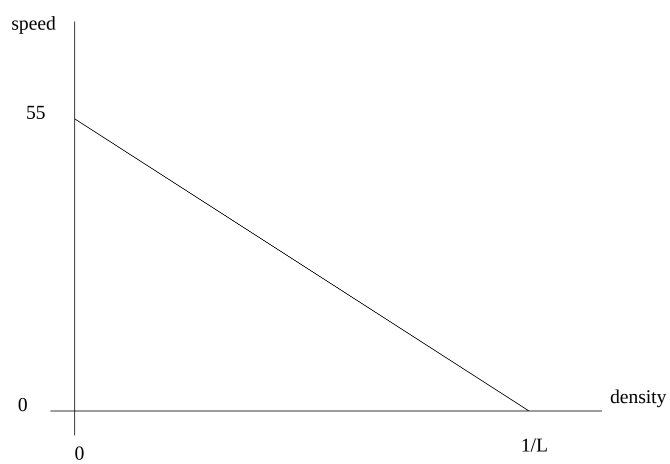
Let $X_j(t)$ be the location of the j'th car at time t, measured in miles along the road, say. The primary assumption that allows us to track the motion of the cars is that *the velocity of the j'th car depends on the distance between this car and the next.* In other words, drivers adjust their speed depending on how crowded the road is just in front of them. The distance to the next car is $d_j(t) = X_{j+1}(t) - X_j(t)$.

The velocity of the j'th car depends on $d_j(t)$ in some way, but it is more usual to re-express it in terms of the *density* of traffic, which might be measured in cars/mile. The symbol ρ is usually used for density, and can vary between $\rho = 0$ (empty road) and $\rho = 1/L$ where L is the length of a car (in miles!) so that $\rho = 1/L$ corresponds to bumper-to-bumper traffic.

Then the density seen by the j'th driver is

$$\rho_j(t) = \frac{1}{X_{j+1}(t) - X_j(t)}.$$

Now we will suppose the velocity of traffic is given by some function of the density, $U(\rho)$ is the velocity in miles/hour. A simple example might be the linear relation shown below. The velocity is 0 when $\rho = 1/L$ (bumper-to-bumper) and increases to some maximum value, say 55 miles/hour, as the density decreases.



The motion of the j'th car is then determined by a *differential equation*,

$$\frac{d}{dt} X_j(t) = U(\rho_j(t)).$$

This can be approximated by taking discrete time steps of length Δt and moving each car a little bit over each time step based on the density seen at the corresponding time. Let X_j^n denote the approximate location of the j'th car at time $t_n = n\Delta t$. Then we can update the car positions by

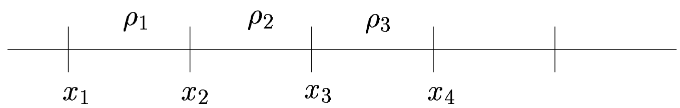
$$X_j^{n+1} = X_j^n + \Delta t U(\rho_j^n)$$

where the density seen by the j'th car at time t_n is $\rho_j^n = 1/(X_{j+1}^n - X_j^n)$. This is the approach used to generate the animation seen on the web page.

Second approach — A finite volume method for density

In modeling the flow of air it is not possible to keep track of how every molecule moves, since there are on the order of 10^{23} molecules in any region of interest. The approach usually used is to split the region up into small pieces (finite volumes) and keep track of the density in each region.

In the case of traffic flow, this amounts to subdividing the highway into short stretches of highway as indicated below:



Each stretch (grid cell) has length Δx (in miles) and if ρ_i^n is an estimate of the density in the i'th cell at time t_n (in cars/mile), then the number of cars in the i'th cell is roughly $\Delta x \rho_i^n$.

How many cars will be on this stretch of highway at time t_{n+1}? Assuming cars don't vanish or appear spontaneously, the number will change only becuase of cars entering the cell at x_i or leaving at x_{i+1}. The flow of cars past a given point per unit time is called the *flux* of cars. This can be computed as the product of the density (cars/mile) times the speed (miles/hour) giving a flux in cars/hour:

$$\text{flux} = F = \rho u.$$

The flux of cars passing the point x_i is approximately

$$F_i^n = \rho_{i-1}^n u_{i-1}^n,$$

based on the density in the cell to the left, from which cars are arriving at x_i.

Over a short time period Δt, the number of cars passing x_i will be approximately $\Delta t F_i^n$, so the total number of cars in this cell at time t_{n+1} can be approximated by

$$\begin{aligned} \Delta x \rho_i^{n+1} &= \Delta x \rho_i^n + \Delta t F_i^n - \Delta t F_{i+1}^n \\ &= \Delta x \rho_i^n + \Delta t(\rho_{i-1}^n u_{i-1}^n - \rho_i^n u_i^n). \end{aligned}$$

This gives a formula for updating the density:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}(\rho_i^n u_i^n - \rho_{i-1}^n u_{i-1}^n).$$

In computing this, we use the relation $u = U(\rho)$ to calculate the velocity based on the density in each cell, *e.g.*, $u_i^n = U(\rho_i^n)$.

Note that we can rewrite this formula as

$$\left(\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t}\right) + \left(\frac{\rho_i^n u_i^n - \rho_{i-1}^n u_{i-1}^n}{\Delta x}\right) = 0.$$

If we let Δx and Δt go to zero, this suggests a *partial differential equation*:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}(\rho(x,t)u(x,t)) = 0.$$

Partial differential equations of this sort are the basic tool for studying fluid dynamics.

Here's how the density evolves with this sort of finite volume method. (Actually a more complicated method is used which gives "sharper" results.) Each circle represents one value ρ_i^n. In this calculation there were 100 cells. Here the density function was rescaled so that $\rho = 1$ corresponds to bumper-to-bumper traffic, *i.e.*, the density is measured in "cars per car length".

Can you figure out how cars are moving from these density plots?