Exam II: solutions

AMATH351

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1. (**20 points**) For the following equations, write down the form of the particular solution, using the method of undetermined coefficients. You do not have to find the value of the coefficients.

   a) $y'' + y = xe^{-x} + \cos(x)$

   b) $y'' + y = (10x^5 - x^3 + 23x^2 - x - 17)e^x \cos(6x)$

   First, we have to solve the homogeneous part, which is the same for both problems. The characteristic equation is $\lambda^2 + 1 = 0$, giving $\lambda_{1,2} = \pm i$, and thus $y_h = c_1 \cos(x) + c_2 \sin(x)$. Now we use the method of undetermined coefficients.

   a) $y_p = (Ax + B)e^{-x} + x(C\cos(x) + D\sin(x))$. We have multiplied the second part of the guess by x, because parts of this guess are contained in the homogeneous solutions.

   b) $y_p = e^x\Big((Ax^5+Bx^4+Cx^3+Dx^2+Ex+F)\cos(6x)+(Gx^5+Hx^4+Ix^3+Jx^2+Kx+L)\sin(6x)\Big)$.

2. (**15 points**) Solve
$$y' + xy = 0, \quad y(0) = 1,$$
   using series methods.

   Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$. Also, $xy = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{j=1}^{\infty} a_{j-1}x^j = \sum_{n=1}^{\infty} a_{n-1}x^n$. Now we plug all of this into the equation, and group different powers of x^n: $\sum_{n=1}^{\infty} x^n \left(a_{n+1}(n+1) + a_{n-1}\right) + a_1 = 0$. From this $a_1 = 0$ and $a_{n+1} = -a_{n-1}/(n+1)$, for $n = 1, 2, 3, \ldots$. This gives $a_2 = -a_0/2$, $a_3 = 0$, $a_4 = -a_2/4 = a_0/2 \cdot 4$, $a_5 = 0$, $a_6 = -a_4/6 = -a_0/2 \cdot 4 \cdot 6$, etc. Thus we obtain
$$y = a_0 - \frac{a_0}{2}x^2 + \frac{a_0}{2 \cdot 4}x^4 - \frac{a_0}{2 \cdot 4 \cdot 6}x^6 + \ldots.$$
   Using the initial condition, we get $a_0 = 1$. Our solution can be rewritten as
$$y = 1 - \left(\frac{x^2}{2}\right) + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 - \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 + \ldots = e^{-x^2/2}.$$

3. (**20 points**) Solve the initial-value problem
$$\begin{cases} y'' - 4y' + 4y = e^{2t}, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

This is the same equation as the first problem on the practice test. Its general solution is (see on-line):

$$y = c_1 e^{2t} + c_2 t e^{2t} + \frac{t^2}{2} e^{2t}.$$

Then $y' = 2c_1 e^{2t} + 2c_2 t e^{2t} + c_2 e^{2t} + t^2 e^{2t} + t e^{2t}$. Using the initial conditions gives $0 = c_1$ and $1 = 2c_1 + c_2$, so that $c_1 = 0$, $c_2 = 1$. Thus the final solution is $y = te^{2t} + \frac{t^2}{2} e^{2t}$.

4. (**25 points**) Consider the equation

$$x^2 y'' - x(x+2)y' + (x+2)y = 2x^3.$$

   a) Check that $y_1 = x$ is a solution of the homogeneous equation.

      In order to plug this in, we have $y_1' = 1$, $y_1'' = 0$. Thus $0 - x(x+1) + (x+2)x = 0$ and $y_1 = x$ is a solution of the homogeneous equation.

   b) Find a second linearly independent solution of the homogeneous equation.

      We use $y_2 = y_1 \int W/y_1^2 dx$, where W is the Wronskian, obtained from Abel's formula: $W = c \exp(-\int p(x) dx)$. Here $p(x) = -x(x+2)/x^2 = -1 - 2/x$. Thus $W = c \exp(\int (1 + 2/x) dx) = ce^{x + 2\ln(x)} = ce^x x^2 = cx^2 e^x$. Using this gives $y_2 = x \int (ce^x x^2/x^2) dx = cx \int e^x dx = cxe^x = xe^x$, choosing $c = 1$.

   c) Having found $y_2 = xe^x$ above, find a particular solution of the equation.

      We have to use variation of parameters, since the homogeneous part does not have constant coefficients. We have

$$y_p = -y_1 \int \frac{g y_2}{W} dx + y_2 \int \frac{g y_1}{W} dx.$$

      Here $y_1 = x$, $y_2 = xe^x$, $W = x^2 e^x$ (from part b), and $g = 2x^3/x^2 = 2x$. Thus

$$\begin{aligned} y_p &= -x \int \frac{2x\ xe^x}{x^2 e^x} dx + xe^x \int \frac{2x\ x}{x^2 e^x} dx \\ &= -2x \int dx + 2xe^x \int e^{-x} dx \\ &= -2x\ x + 2xe^x\ (-e^{-x}) \\ &= -2x^2 - 2x. \end{aligned}$$

   d) Write down the general solution.

      The general solution is $y = c_1 y_1 + c_2 y_2 + y_p = c_1 x + c_2 xe^x - 2x^2 - 2x$.

5. (**20 points**) The spring + shock absorber on your Harley-Davidson motorcycle can be modeled using the differential equation $mu'' + cu' + ku = F(t)$. Suppose that the Harley's mass is 400 kg and the spring is designed so that the natural frequency is 500 sec^{-1}.

a) Your shock absorber is broken, so there is no damping. As you cruise down the road at $v = 100$ kph (kilometers per hour), you hit a "washboarded" section. The washboards result in a force $F(t) = F_0 \cos(\omega t)$, where $\omega = 2\pi v/L$, where L is the period of the washboard. For what L (measured in meters) is the effect on the Harley the most dangerous? Explain.

The most dangerous effect is resonance, because the amplitude of the oscillation becomes unbounded. For resonance, we need $\omega = \omega_0$, where $\omega_0 = 500/\text{sec}$. Thus $500 = 2\pi v/L$. Solving for L we get $L = 2\pi v/500 = 2\pi 100 * (1000/3600)/500 = pi/9$ meters, or $L \approx 0.3$ meters.

b) You take your Harley to the shop to have it fixed. To order the proper shock absorber, you must find the required coefficient of damping, c, so that the system is at least critically damped. What's the smallest allowable value of c (expressed in kg/sec)?

To have critical damping, we need $c^2 = 4km$. We don't know k, but $\omega_0^2 = k/m$, so $k = m\omega_0^2$, and thus $c^2 = 4m^2\omega_0^2$, or $c = 2m\omega_0 = 2\ 400\ 500 = 4\ 10^5$ kg/sec.

6. (**Extra credit: 20 points**) Describe in detail how you would go about solving the n-th order nonhomogeneous constant coefficient equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = F(x).$$

Discuss different cases if necessary. Provide as much detail as you can.

This is an open ended question. Let's see. First, we realize that the general solution will be a superposition of solutions of the homogenous equation and a particular solution.

- **Homogeneous:** In analogy to second-order equations, we can guess solutions of the form $y = e^{\lambda x}$. This gives the characteristic equation $a_n\lambda^n + \ldots + a_1\lambda + a_0 = 0$. In general this equation has n solutions. These can be complex, in which case they will be complex conjugates. For real λ_k, we get $y_k = e^{\lambda_k x}$ is a solution. If λ_k occurs multiple time (say, multiplicity N_k), then $x^j e^{\lambda_k x}$, $j = 0, 1, \ldots, N_k - 1$ are all solutions as well. For complex $\lambda_k = \alpha_k + i\beta_k$, we have $e^{\alpha_k x}\cos(\beta_k x)$ and $e^{\alpha_k x}\cos(\beta_k x)$ are both solutions. If complex λ_k occur with higher multiplicity, then we need to multiply by powers of x, as in the real case.
- **Particular:** For the particular solution, we can use the method of undetermined coefficients if $F(x)$ is a product of exponentials, cosines, sines, and polynomials, and sums thereof. Otherwise we have to use variation of parameters. A formula similar to the one for second order equations can be derived, but requires quite a bit of work.