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1 Introduction

The overall objective of our FRG has been to understand the dynamics of surface water waves, where the waves are three-dimensional, have finite amplitudes, and propagate in water of arbitrary depth. With a deeper understanding of the dynamics, we hope to construct appropriate building blocks of a practical theory of water waves, without the limitations inherent in other theories. In this third year of our FRG, we believe that we have achieved major success in two areas:

i) the dynamics of waves in deep water, and
ii) a numerical method to compute the stability of stationary wave profiles rapidly and accurately.

We report on each of these areas below. We have also made progress in studies of waves in shallow water, but we claim no breakthroughs there. At the end of our third year, some of us have depleted our FRG funds and will write separate Final Reports in the near future. Those of us with unexpended funds have requested no-cost extensions to our grants, and will write Final Reports next year.

Our work was disrupted when our friend and co-worker, Joe Hammack, passed away unexpectedly in September 2004. Joe was responsible for much of the experimental work that unites the projects in our FRG. We miss him.

2 Scientific results

2.1 Dynamics of waves in deep water

The focus of our work on waves in deep water is to explain the dynamics of the apparently stable pattern of surface waves propagating in deep water, shown in Figure 1. This wave pattern is one of several that we produced in the wave tank shown in the figure, and many of these patterns seem to propagate stably in deep water. See Hammack *et al.* (2005) for a detailed discussion of two-dimensional patterns of surface waves in deep water, and find more photos and videos of wave patterns like these at our FRG website,

http://www.amath.washington.edu/~bernard/frg_surfacewaves.html

under "Experiments", then click "Waves with 2d patterns in deep water."

The two-dimensional, bi-periodic structure of the wave pattern in Figure 1 is no surprise - it was fed into the waves through the array of 32 individually programmable paddles shown at the top of the figure. What did surprise us was the apparent stability of this wave pattern as it propagated in deep water. Its stability seems to contradict a basic principle of nonlinear wave propagation, established by the pioneering work of Benjamin & Feir

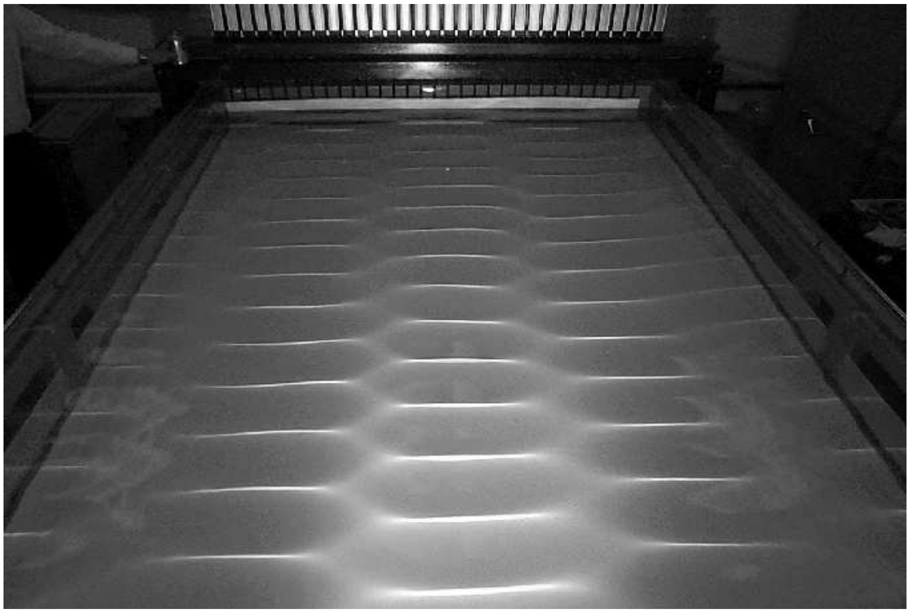

Figure 1: An oblique interaction of two one-dimensional wavetrains in deep water produces this two-dimensional pattern, which propagates with nearly permanent form away from the array of paddles at the top of the picture, with no evidence of instability within the test section of the wave tank.

(1967) and Zakharov (1968). They showed that a uniform train of plane, monochromatic waves of finite amplitude is unstable in deep water without dissipation. Other work about the same time (*e.g.*, Lighthill, 1965) showed that this kind of instability is general. It is called a "modulational" or "Benjamin-Feir" instability, and it affects uniform trains of finite-amplitude waves in plasma physics, in optics and elsewhere.

A uniform train of plane, monochromatic (*i.e.*, same frequency) waves of finite amplitude is fundamental to many theories of wave propagation, so the discovery of the modulational instability sparked a search for other wave forms that might be stable in deep water. The search was unsuccessful: no wave form was found that could propagate stably on a two-dimensional surface of deep water without dissipation. In this context, the apparently stable wave pattern shown in Figure 1 forced us to reconsider the stability of surface waves in deep water.

A partial resolution of the paradox was found by Segur *et al.* (2005), in work supported by this grant. In Zakharov's important 1968 paper, he derived the two-dimensional nonlinear Schrödinger equation (NLS),

$$i\partial_t A + \alpha \partial_x^2 A + \beta \partial_y^2 A + \xi |A|^2 A = 0, \tag{1}$$

to describe the slow evolution of a nearly monochromatic train of plane waves of finite amplitude in deep water without dissipation, where $\{\alpha, \beta, \xi\}$ are known real numbers with $\alpha\beta < 0, \quad \alpha\xi > 0$. In this model, a uniform train of plane waves with finite amplitude in deep water corresponds to a solution of (1) with no spatial dependence (*i.e.*, $\partial_x A = \partial_y A = 0$), and Zakharov showed that such a solution of (1) is necessarily unstable. Physical experiments on waves in deep water always exhibit wave damping, so several authors (*e.g.*, Lake *et al.*, 1977) generalized (1) by including an empirical factor for dissipation:

$$i\partial_t A + \alpha \partial_x^2 A + \beta \partial_y^2 A + \xi |A|^2 A + i\delta A = 0. \tag{2}$$

Segur *et al.* (2005) noticed what earlier workers had missed: *any amount of damping (i.e, any $\delta > 0$ in (2)) stabilizes the Benjamin-Feir instability.* That paper proves that the solution of (2) corresponding to a spatially uniform train of plane waves with finite amplitude is both linearly and nonlinearly stable (to small perturbations in the initial data). In addition, their work shows that (2) predicts the evolution of wave profiles measured in laboratory experiments much more accurately than earlier theories (without dissipation) had done. Thus even when dissipation is weak, it can have an important dynamical effect by stabilizing the Benjamin-Feir instability. We consider this finding one of the (two) major results of our FRG.

We regard this result as important precisely because it forces a re-evaluation of the Benjamin-Feir instability, a basic concept in nonlinear wave dynamics. The result has been controversial, probably for the same reason, despite the good agreement between the dissipative theory and experiments.

In related work under this grant, Canney & Carter (preprint) have studied the effect of adding wave damping to various generalizations of the nonlinear Schrödinger equation, including two "broader bandwidth" models studied by Dysthe (1979) and Trulsen & Dysthe (1996). These models are known to exhibit modulational instabilities, and Canney & Carter showed that adding a damping term to each of these models stabilizes their instabilities, just as it does for (1). The significance of their result is that it shows that the stabilizing effect of wave damping is more general, and is not restricted to (1) and (2).

All of the work on wave damping described above deals with trains of plane waves, which have one-dimensional surface patterns. But the work was inspired by two-dimensional surface patterns like that shown in Figure 1, so it remains to apply these concepts to models of two-dimensional wave patterns. Hammack *et al.* (2005) had derived a model of two coupled NLS equations to describe the evolution of two intersecting wave trains in deep water without dissipation. When one adds wave damping to this two-equation model, it takes the (somewhat intimidating) form:

$$\begin{aligned} i(\partial_x A + c_1\partial_t A + d_1\partial_y A) + \epsilon[\alpha_1\partial_t^2 A + \beta_1\partial_y^2 A + \gamma_1\partial_t\partial_y A + \xi_1|A|^2A + \zeta|B|^2A] + i\delta_1 A = 0, \\ i(\partial_x B + c_2\partial_t B + d_2\partial_y B) + \epsilon[\alpha_2\partial_t^2 B + \beta_2\partial_y^2 B + \gamma_2\partial_t\partial_y B + \xi_2|B|^2B + \zeta|A|^2B] + i\delta_2 B = 0. \end{aligned} \tag{3}$$

In recent work under this grant, M. Bleymaier showed that with $\delta_1 = \delta_2 = 0$, (3) exhibits a modulational instability, analogous to that in (1). She also showed that for $\delta_1 > 0$, $\delta_2 > 0$, dissipation stabilizes the instability. Experiments to apply her theory to wave patterns like that in Figure 1 are now under way. We hope to have definitive comparisons within the next few months.

What is the *broader impact* of this work on wave damping?

1. Within oceanography, the Benjamin-Feir instability is built into many operational models of wave forecasting. (Google WAMDI for examples.) The work described above shows that the Benjamin-Feir instability needs to be re-evaluated because of the effect of damping. It is too early to tell how such re-evaluations might change the operational models.

2. The nonlinear Schrödinger equation, (1), is an approximate model not only of surface waves in deep water, but also electromagnetic waves in an optical fiber (*e.g.*, Hasegawa & Kodama, 1995), and spin waves in a thin magnetic film (Wu *et al.*, 2004). In all of these physical contexts, wave damping exists that is omitted from the NLS model.

The work on waves in deep water, described here, was presented at a recent conference on nonlinear waves

http://math.uccs.edu/~soliton/

After the presentation, S. K. Turitsyn expressed strong interest in seeing how wave damping might help (instead of just hurt) to send signals down an optical fiber. Similarly, C. E. Patton indicated that he would like to explore how damping might be used to advantage in controlling spin waves in a magnetic film. In this way, our results on the stability of waves in deep water can influence research on wave propagation in scientific fields quite far from water waves.

3. Bose-Einstein condensates (BECs) are modeled by an equation closely related to NLS, the Gross-Pitaevskii equation (GP). After the presentation of our work at a IMACS conference in Athens GA, M. Ueda discussed with us the effects of damping in BECs. One type of damping in BECs is modeled with nonlinear damping terms, but Ueda found that the small damping effect modeled by the same type of damping we have found necessary in NLS is also vital in the phenomenon he was studying in BECs. This linear damping must be included in the GP equation in order to simulate stable vortex patterns in numerical simulations that are observed in experiments. Without this term and its small effect, the vortices fly apart. This is an example in another field in which this type of damping stabilizes patterns.

2.2 Numerical computation of stability

During our search for an explanation of the apparently stable waveforms shown in Figure 1, we studied several effects that we thought might stabilize the known instability. These studies typically led to a determination of linear stability by solving a set of linear differential equations containing an eigenvalue (that expresses the growth rate of the possible instability). Solution of these equations often required a combination of analysis and numerical computation (*cf.* Carter & Segur, 2003). In some cases, earlier numerical solutions of these problems were already in print, with conflicting results. These and other stability problems demonstrated the need for a fast, accurate numerical method to compute the spectra of linear operators, and that could be applied to a variety of problems.

Recent work by Deconinck & Kutz (preprint), supported in part by this grant, seems to have found such a method. After developing their "new" method, Deconinck & Kutz realized that its essential ideas where due to Hill (1886) in his pioneering study of motions in the solar system. In a sense, what they actually discovered was that Hill's method can easily be generalized beyond the study of what we now call "Hill's equation," and that the method computes (approximately, but very accurately) the spectrum of linear differential operators much faster than other competing methods, such as finite difference methods. One reason for this extra speed is that computational errors due to truncation decrease exponentially instead of just algebraically. The method has been used by different authors in different communities (see the preprint of Deconinck & Kutz) for some time, without those authors apparently realizing the generality of the method they employed. Not realizing this generality often led to an incorrect use of the method. The method is also known as Bloch's method in solid state physics, and in this incarnation has been used by Craig and Nicholls to determine numerically the stability of spatially periodic stationary wave patterns of the Euler equations (in progress). Deconinck, Nicholls and Thelwell are continuing this work.

The method is especially well suited to computing the spectrum of a linear differential operator with periodic coefficients. For such a problem, the method builds in elements of both Floquet theory and Fourier analysis, so we call it the "Floquet-Fourier-Hill" method. It is a significant improvement over other methods already used in our group, including that of Carter & Segur (2003). Other papers, now in preparation, use this method to compute stability results that were out of reach with other methods. Go to

http://www.amath.washington.edu/~bernard/frg/papers.html

for a preprint.

What is the broader impact of this faster numerical method to compute spectra of linear operators? Our results are new, and there is always a danger of over-selling the potential benefits of a new discovery. Subject to that caveat, the potential implications of this new method seem very large, because questions of stability lie at the heart of many scientific problems, including confinement and control of a thermonuclear plasma (Hazeltine & Meiss, 1992), confinement and control of a Bose-Einstein condensate (Anderson *et al.*, 1995), propagation of signals in an optical fiber (Hasegawa & Kodama, 1995), propagation of spin waves in a magnetic film (Wu *et al.*, 2003), and more.

2.3 Solitary wave interactions in shallow water

Scott Russell (1838) first identified solitary waves experimentally, and he studied their interactions in shallow water. The subject enjoyed renewed interest after Kruskal & Zabusky (1965) defined "solitons" to describe some solitary waves. In two papers supported by this grant, Hammack *et al.* (preprint) and Craig *et al.* (preprint) carried out high-resolution experiments on colliding solitary waves in shallow water, and compared their experimental results with numerical integration of Euler's equations. They studied both counter-propagating (*i.e.*, head-on) collisions and co-propagating (following) collisions. One nice result of their study was that analytical results by Lax (1968), predicting that solitary wave interactions can be qualitatively different in different parameter regimes, were confirmed by numerical solution of Euler's equations with a small shift in the boundaries of the parameter regimes. The experiments were conducted in a parameter regime within this shift and agreed with the predictions from the simulations using Euler's equations.

3 Dissemination of results

3.1 Scientific papers

The most long-lasting method to disseminate scientific results is with published papers. Recent papers supported by this FRG are listed in section 4, below.

3.2 Workshop at the Fields Institute

The original plan of this FRG contained a provision to organize and run a Workshop on Free Surface Water Waves, to be held after the second year of the FRG. This workshop was held at the Fields Institute for Research in Mathematical Sciences, in Toronto Canada, during June 14-18, 2004. It was jointly sponsored by the NSF (through this grant) and by the Fields Institute. It fit naturally into a Thematic Year on Partial Differential Equations, held at the Fields Institute during 2003-2004. See

http://www.fields.utoronto.ca/programs/scientific/03-04/pde/water_waves/

for more information about the workshop program.

The workshop brought together an international mix of about fifty scientists, including mathematicians, engineers and oceanographers, to consider issues related to the dynamics of ocean waves. Theoreticians included analysts intrigued by the inherent difficulty of free-boundary problems, computationalists who seek to predict the detailed dynamics of ocean waves with accurate numerical codes, and operational forecasters, concerned with developing computational models that can accurately forecast wave conditions in the open ocean. Experimentalists included researchers using laboratory facilities and others conducting field observations in the open ocean or near shore region. The range of backgrounds of the participants was broad, so the workshop was structured to include plenty of time for informal gatherings, discussion and work. To facilitate discussion among people with disparate research approaches and perspectives, ad hoc "worksessions" were provided in which a participant could reserve a room on a sign-up sheet for informal discussion on a particular topic. These worked quite well.

Several participants in the workshop commented that they were pleased with interactions among groups with very different viewpoints, and they enjoyed the opportunity to discuss problems of common interest within such a varied group. Another common remark was that "The Fields Institute is really a great place," and provided a perfect venue for enabling interactions among disparate groups of scientists. An earlier workshop on water waves had been held at the Newton Institute (in Cambridge, England) in 2001, and several participants recommended holding another workshop on the topic in about 2008.

3.3 Other conferences

More than one member of our FRG presented work at each of the following international conferences.

i) SIAM Conference on Nonlinear Waves and Coherent Structures - Orlando, FL, October 2-4, 2004
    - John Carter, Bernard Deconinck and Harvey Segur each gave lectures.

ii) Fourth IMACS International Conference on Nonlinear Evolution Equations and Wave Phenomena - Athens, GA, April 11-14, 2005
    - John Carter - session organizer, gave a lecture
    - Walter Craig - member of the scientific program committee
    - Bernard Deconinck - session organizer, gave a lecture
    - Philippe Guyenne (postdoc with Walter Craig) - gave a lecture
    - Diane Henderson - gave a lecture
    - Firat Kiyak (undergrad student with Bernard Deconinck) - presented a poster
    - Dave Nicholls - session organizer, gave a lecture
    - Mike Nivala (grad student with Bernard Deconinck) - presented a poster
    - Roger Thelwell (FRG postdoc with Bernard Deconinck) - gave a lecture
    - William Whitwell (undergrad student with John Carter) - presented a poster

iii) International Conference on Nonlinear Waves, Integrable Systems and Their Applications - Colorado Springs, CO, June 4-7, 2005
    - Diane Henderson and Harvey Segur each gave lectures

3.4 Additional lectures

i) John Carter
  - October, 2004 - U. of Washington, Seattle, WA
  - December, 2004 - UNAM, Mexico City, Mexico
  - May, 2005 - Faculty research poster, Seattle U., Seattle, WA
  - Poster presentations by undergrad students
    - N. Canney, Murdock Conference, November, 2004
    - W. Whitwell, NCUR, April, 2005

ii) Bernard Deconinck
  - December, 2004 - Tech. U. Berlin, Berlin, Germany
  - February, 2005 - Simon Fraser U., Vancouver, BC, Canada
  - March, 2005 - U. of Colorado, Boulder, CO
  - March, 2005 - U. of Notre Dame, South Bend, IN

iii) Joe Hammack
  - June, 2004 - Offshore Mechanics and Arctic Engineering: A Symposium to honor Theodore Yao-Tsu Wu, Vancouver, BC, Canada

iv) Diane Henderson
  - January, 2005 - U. of Washington, Seattle, WA
  - March, 2005 - MIT, Cambridge, MA

v) Harvey Segur
  - January, 2005 - 14th Aha Huliko'a Hawaiian Winter Workshop, Honolulu, HI
  - March, 2005 - 97th Distinguished Research Lecture, U. of Colorado, Boulder, CO
  - May, 2005 - Los Alamos CNLS annual conference, Santa Fe, NM

4 Recent work supported by this grant

During 2004-2005, the following papers, supported by this grant, have been published, accepted for publication, or submitted.

- Bradley, R.M., B. Deconinck & J.N. Kutz, 2005, "Exact nonstationary solutions of the mean-field equations of motion for two-component Bose-Einstein condensates in periodic potentials," *J. Phys. A*, **38**, pp. 1901-1916.

- Canney, N.E. & J. D. Carter, "Stability of dissipative plane waves on deep water," submitted for publication.

- Craig, W., P. Guyenne, J. Hammack, D. Henderson & C. Sulem, "Solitary water wave interactions," submitted for publication.

- Deconinck, B., P.G. Kevrekidis, H.E. Nistazakis & D.J. Frantzeskakis, 2004, "Linearly coupled Bose-Einstein condensates: from Rabi oscillations and quasi-periodic solutions to sloshing domain walls and spiral waves," *Phys Rev. A*, **70**, 063605.

- Deconinck, B. & J.N. Kutz, “Computing spectra of linear operators using Hill’s method,” submitted for publication.

- Deconinck, B., W. Hereman, M. Colagrosso, R. Sayers, M. Nivala & M. Hickman, 2005, “Continuous and discrete homotopy operators with application to integrability testing,” to appear in *Symbolic Computation for Differential Equations*, ed. by Dongming Wang.

- Hammack, J.L., D.M. Henderson, P.Guyenne & M. Yi, 2004, “Solitary wave collisions,” *Proc. Offshore Mechanics and Arctic Engineering: A Symposium to honor Theodore Yao-Tsu Wu,* World Scientific Publishers, 10pp..

- Hammack, J.L., D.M. Henderson & H. Segur, 2005, “Progressive waves with persistent two-dimensional surface patterns in deep water,” *J. Fluid Mech.*, **532**, pp.1-51.

- Segur, H., D. Henderson, J. Carter, J. Hammack, C-M Li, D. Pheiff & K. Socha, 2005, “Stabilizing the Benjamin-Feir instability,” to appear in *J. Fluid Mech.*

- Segur, H., D.M. Henderson & J.L. Hammack, “Can the Benjamin-Feir instability spawn a rogue wave?” to appear in *Proceedings of the 14th Aha Huliko’a Hawaiian Winter Workshop.*

5 Other references

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