On Analyticity of Traveling Water Waves

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In this paper we establish the existence and analyticity of periodic solutions of a classical free–boundary model of the evolution of three–dimensional, capillary–gravity waves on the surface of an ideal fluid. The result is achieved through the application of bifurcation theory to a boundary perturbation formulation of the problem, and it yields analyticity jointly with respect to the perturbation parameter and the spatial variables. The traveling waves we find can be interpreted as resulting from the (nonlinear) interaction of two two–dimensional wavetrains, giving rise to a periodic traveling pattern. Our analyticity theorem extends the most sophisticated results known to date in the absence of resonance; "short crested waves," which result from the interaction of two wavetrains with unit amplitude ratio are realized as a special case. Our method of proof also sheds light on the convergence and conditioning properties of classical boundary perturbation methods for the numerical approximation of traveling surface waves. Indeed, we demonstrate that the rather unstable numerical behavior of these approaches can be attributed to the strong but subtle cancellations in the formulas underlying their classical implementations. These observations motivate the derivation and use of an alternative, stable, formulation which, in addition to providing our method of proof, suggests new stabilized implementations of boundary perturbation algorithms.

Keywords: Capillary–gravity water waves, traveling waves, boundary perturbations.

1. Introduction

The stable and accurate numerical simulation of free–surface ocean dynamics is one of the central problems in computational fluid mechanics. From shoaling and breaking of waves over nearshore regions to energy, momentum, and scalar transport in the open ocean, the rapid and reliable approximation of the surface of a fluid is a necessary tool in problems of physical relevance. Surface waves that propagate with constant velocity and without change of form (the traveling waves) are a distinguished class of motions which are believed to be a fundamental building–block of surface ocean dynamics.

In this paper we take up the mathematical question of regularity properties of traveling wave solutions of the classical water wave model (see § 2) which constitutes an accurate representation for the motion of the free surface of the ocean. In particular we demonstrate that the water wave problem in d–dimensions admits *surfaces* of solutions, parameterized by $(d-1)$–many parameters $\varepsilon \in \mathbf{R}^{d-1}$, which are jointly analytic in the parametric and spatial variables. Our method of proof is perturbative in nature and general enough to encompass every case away from resonances (see § 4.b).

The first rigorous existence theorems for traveling wave solutions to the water wave model date to the results in two space dimensions without surface tension by Levi–Civita [LC25] (infinite depth) and Struik [Str26] (finite depth) who used complex variables techniques. With the advent of the modern computer there was a resurgence of interest in the problem in the 1970's and 1980's as highly nonlinear waveforms could now be simulated, see e.g. [Sch74, Rob83, SR83, MR87]. This resurgence was also accompanied by new theoretical developments. For instance, Reeder & Shinbrot [RS81a, RS81b] studied the phenomena of Wilton ripples which arise in two–dimensional traveling capillary–gravity water waves. They showed existence and smoothness of branches of traveling wave solutions which exist in the presence of resonance in the linearized problem. Other important theoretical results in two space dimensions include those of Toland and collaborators who used various integral formulations of the two–dimensional water wave problem coupled with variational techniques (e.g. minimizers, mountain pass). An important early result [Tol78] of Toland's established the *global* existence of the bifurcating branch of solutions all the way to the Stokes singularity. Jones & Toland [JT86] also looked at surface tension effects in two dimensions, and subharmonic bifurcations in [JT85]. Subharmonic bifurcation was also the object of Buffoni, Dancer, & Toland [BDT00b], who have recently studied analyticity properties of two–dimensional traveling waves in [BDT00a].

In three dimensions, on the other hand, the most general results to date are those of Craig & Nicholls [CN00] who, in the presence of non–zero surface tension, established existence of traveling capillary–gravity water waves with arbitrary fundamental period. The theorem of Craig & Nicholls used the surface formulation of Zakharov [Zak68] and Craig & Sulem [CS93] coupled with the Lyapunov–Schmidt procedure from bifurcation theory. Other results in three dimensions include that of Sun [Sun93] who viewed the traveling wave as generated by a surface pressure, and Groves & Mielke [GM01] and Groves [Gro01] who have studied traveling waves using a "spatial dynamics" approach. In this formulation the direction of propagation, in the traveling wave equations, is considered the dynamical quantity; the transverse direction is typically considered to be periodic and then periodic (in propagation direction) solutions are sought.

The three–dimensional results most closely related to those we present herein are those of Reeder & Shinbrot [RS81c] who demonstrate the existence of "short–crested" capillary–gravity waves of sufficiently small amplitude; short–crested waves are typically defined [DK99] as the waves which result from the (nonlinear) interaction of two periodic wavetrains of equal amplitude, infinite extent, and non–zero angle of interaction (as the angle of interaction approaches zero the waves are typically referred to as "long–crested"). Akin to the method we adopt in § 3, Reeder & Shinbrot also use a "domain flattening" change of variables. Our results expand on those of [RS81c] in that we allow for the interaction of wavetrains of *arbitrary* amplitude ratio, i.e. not necessarily short–crested waves. To this end, our approach entails the use of *multi–dimensional* perturbation parameters, leading to significantly more complex recursions and estimates (see § 4).

In addition to establishing existence and analyticity of hypersurfaces of traveling water waves, our work also sheds light on the convergence and conditioning properties of classical boundary perturbation methods for the numerical simulation of traveling capillary–gravity water waves. In particular we discuss the method of

Stokes [Sto47] (which we term the method of "Field Expansions" (FE)) that was further refined and carried out to high order by Roberts [Rob83], Schwartz & Roberts [SR83], and Marchant & Roberts [MR87] for three–dimensional traveling water waves in the absence of surface tension. As we show in § 2.d this method produces *unstable* results as the perturbation order is increased due to subtle but significant *cancellations* which are present in the underlying recursions. As we explain, a further consequence of this observation is that these FE recursions cannot be used for a direct proof of existence or analyticity. However, as we anticipated above, a direct proof can be realized once a "domain flattening" change of variables is effected as this can be shown to implicitly account for all significant cancellations. This latter fact suggests a stabilized approach to numerical simulation, whose thorough investigation we defer to future work (see also [NR01a, NR01b, NR03a, NR03b, NR03c]).

The remainder of the paper is organized as follows: first, in § 2 we introduce the equations of motion and the classical "Field Expansions" approach to simulating traveling water waves; in particular, in § 2.d we demonstrate how these classical recursions rely heavily on significant cancellations for their convergence. In § 3 we introduce a change of variables which substantially ameliorates these cancellations and paves the way for the analyticity proof of § 4; some auxiliary results necessary for this latter proof are collected in Appendix A.

2. Preliminaries

In this section we briefly review the equations of motion of the water wave problem (ideal fluid, free–surface flow) and outline the classical "Field Expansion" (FE) technique for perturbatively computing solutions. With the aid of a numerical implementation of this algorithm and several simulations, we illustrate how these recursions are inherently unstable at high orders due to underlying *cancellations*.

(*a*) *Equations of Motion*

Consider a d–dimensional ($d = 2, 3$) fluid (one vertical dimension specified by the variable y and $(d-1)$ horizontal dimensions specified by x) bounded below by an impermeable bottom at $y = -h$ (h possibly infinite) and above by an undetermined air/fluid interface, $y = \eta(x, t)$, which occupies the domain

$$S_{h,\eta} = \{(x, y) \in \mathbf{R}^{d-1} \times \mathbf{R} \mid -h < y < \eta(x, t)\}.$$

In the case of finite depth no generality is lost if h is set to one as this simply amounts to a rescaling of independent variables. Consider also the classical assumption that the fluid be periodic with respect to the lattice $\Gamma \subset \mathbf{R}^{d-1}$ which defines a parallelogram of periodicity $P(\Gamma)$. The equations of motion of an ideal fluid in such a domain under the effects of gravity and capillarity are [Lam93]

$$\Delta\varphi = 0 \qquad \text{in } S_{1,\eta} \tag{2.1a}$$

$$\partial_y\varphi(x, -1) = \int_{P(\Gamma)} \partial_y\varphi(x, -1)\, dx, \quad \int_{P(\Gamma)} \varphi(x, -1)\, dx = 0 \tag{2.1b}$$

$$\partial_t\varphi + \tfrac{1}{2}\left|\nabla\varphi\right|^2 + g\eta - \sigma\kappa(\nabla_x\eta) = 0 \qquad \text{at } y = \eta \tag{2.1c}$$

$$-\partial_t\eta - \nabla_x\eta \cdot \nabla_x\varphi + \partial_y\varphi = 0 \qquad \text{at } y = \eta, \tag{2.1d}$$

where φ is the velocity potential, g is the constant of gravity, σ is the constant of capillarity, and κ is the curvature

$$\kappa(\nabla_x \eta) = \text{div}_x \left[\frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}} \right].$$

As we stated, we shall be concerned with traveling waves translating uniformly with speed $c \in \mathbf{R}^{d-1}$, which satisfy

$$\Delta\varphi = 0 \qquad \text{in } S_{1,\eta} \tag{2.2a}$$

$$\partial_y \varphi(x,-1) = \int_{P(\Gamma)} \partial_y \varphi(x,-1)\, dx, \quad \int_{P(\Gamma)} \varphi(x,-1)\, dx = 0 \tag{2.2b}$$

$$[c \cdot \nabla_x]\, \varphi + \tfrac{1}{2} |\nabla\varphi|^2 + g\eta - \sigma\kappa(\nabla_x \eta) = 0 \qquad \text{at } y = \eta \tag{2.2c}$$

$$-[c \cdot \nabla_x]\, \eta - \nabla_x \eta \cdot \nabla_x \varphi + \partial_y \varphi = 0 \qquad \text{at } y = \eta. \tag{2.2d}$$

(b) *Bifurcation Theory*

Adopting a bifurcation theoretic approach, we seek solutions of (2.2) near the quiescent state ($\varphi = \eta = 0$ and *any* velocity c) which forms a "trivial" family of solutions. Bifurcation theory requires the analysis of the linearization of (2.2) about these trivial solutions which leads to consideration of the problem

$$\Delta\varphi_1(x,y) = 0 \qquad \text{in } S_{1,0} \tag{2.3a}$$

$$\partial_y \varphi_1(x,-1) = \int_{P(\Gamma)} \partial_y \varphi_1(x,-1)\, dx, \quad \int_{P(\Gamma)} \varphi_1(x,-1)\, dx = 0 \tag{2.3b}$$

$$[c_0 \cdot \nabla_x]\, \varphi_1(x,0) + [g - \sigma\Delta_x]\, \eta_1(x) = 0 \tag{2.3c}$$

$$-[c_0 \cdot \nabla_x]\, \eta_1(x) + \partial_y \varphi_1(x,0) = 0. \tag{2.3d}$$

The periodic boundary conditions, (2.3a), and (2.3b) imply that

$$\varphi_1(x,y) = \sum_{k \in \Gamma', k \neq 0} a_{1,k} \frac{\cosh(|k|\,(y+1))}{\cosh(|k|)} e^{ik\cdot x}, \quad \eta_1(x) = \sum_{k \in \Gamma', k \neq 0} d_{1,k} e^{ik\cdot x}.$$

Equations (2.3c) & (2.3d) become

$$A(c_0,k) \begin{pmatrix} a_{1,k} \\ d_{1,k} \end{pmatrix} \equiv \begin{pmatrix} c_0 \cdot ik & g + \sigma |k|^2 \\ |k| \tanh(|k|) & -c_0 \cdot ik \end{pmatrix} \begin{pmatrix} a_{1,k} \\ d_{1,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.4}$$

for every $k \in \Gamma'$, $k \neq 0$. Thus a non–trivial solution can exist only if the matrix $A(c_0,k)$ is singular for some $k \in \Gamma'$, that is, if the determinant

$$\Lambda_\sigma(c_0,k) = (c_0 \cdot k)^2 - (g + \sigma |k|^2)\, |k| \tanh(|k|) \tag{2.5}$$

vanishes. In this case, if a pair (c_0, k) satisfies $\Lambda_\sigma(c_0, k) = 0$, then a non–trivial solution of (2.3) is

$$\eta_1(x) = \alpha_k(c_0 \cdot k)e^{ik\cdot x} + \bar{\alpha}_k(c_0 \cdot k)e^{-ik\cdot x} \tag{2.6a}$$

$$\begin{aligned}\varphi_1(x, y) &= \alpha_k i(g + \sigma |k|^2)\frac{\cosh(|k|(y+1))}{\cosh(|k|)}e^{ik\cdot x} \\ &\quad - \bar{\alpha}_k i(g + \sigma |k|^2)\frac{\cosh(|k|(y+1))}{\cosh(|k|)}e^{-ik\cdot x},\end{aligned} \tag{2.6b}$$

where $\alpha_k \in \mathbf{C}$ is an arbitrary constant.

Our approach to finding non-trivial solutions of (2.3) when $d = 2$ is to choose a wavenumber $\kappa_1 \in \Gamma'$ ($\kappa_1 \neq 0$) and solve for the corresponding c_0 such that $\Lambda_\sigma(c_0, \kappa_1) = 0$, i.e.

$$c_0 = \pm\sqrt{\frac{(g + \sigma |\kappa_1|^2)\tanh(|\kappa_1|)}{|\kappa_1|}}; \tag{2.7}$$

without loss of generality, we can always select the positive root. With c_0 chosen in this way we can write $\Lambda_\sigma(c_0, k) = k^2(\psi(\kappa_1) - \psi(k))$ where

$$\psi(k) = \frac{(g + \sigma k^2)\tanh(k)}{k}$$

and we have used the fact that $\tanh(k)/k$ is even to drop the absolute value.

Clearly, when $k = 0, \pm\kappa_1$, Λ_σ is zero and the null space of the linearized operator is at least three–dimensional. An important question is whether other wavenumbers will produce zeros of Λ_σ resulting in a higher dimensional null space; this scenario is one of *resonance* and is left outside the scope of our current theory. However, one can easily make some general statements concerning the possibility of resonance. In the case of zero surface tension ($\sigma = 0$)

$$\psi(k) = \frac{g\tanh(k)}{k}$$

which (for $k > 0$) is strictly decreasing, implying that $\psi(k) = \psi(\kappa_1)$ if and only if $k = \pm\kappa_1$. Thus, in this case, there can be no resonance. However, for $\sigma > 0$ the derivative of ψ may vanish at a point if σ/g is sufficiently small. However, even in this case, the existence of an additional *integer* root of $\psi(k) = \psi(\kappa_1)$ will not occur generically.

Our approach to finding non-trivial solutions of (2.3) when $d > 2$ is to choose $(d-1)$ many wavenumbers $\kappa_1, \kappa_2, \ldots, \kappa_{d-1} \in \Gamma'$ ($\kappa_j \neq 0$) and solve the corresponding set of $(d-1)$ equations $\Lambda_\sigma(c_0, \kappa_j) = 0$, i.e.

$$Kc_0 = R \tag{2.8}$$

where $K \in \mathbf{R}^{(d-1)\times(d-1)}$ has rows $\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$, and $R \in \mathbf{R}^{d-1}$ has j-th entry

$$R_j = \pm\sqrt{(g + \sigma |\kappa_j|^2)\,|\kappa_j|\tanh(|\kappa_j|)}. \tag{2.9}$$

Among the 2^{d-1} choices for the vector R we will always choose the one such that $R_j > 0$. When $d > 2$ there is always the possibility, though rare, that additional "resonant" wavenumbers $\kappa_d, \ldots, \kappa_p$ may exist such that $\Lambda(c_0, \kappa_j) = 0$ for $j = d, \ldots, p$. In fact, when $\sigma = 0$ the number p can be infinite; see e.g. [CN00] for a more complete discussion of these issues.

(c) Field Expansions

A classical approach to finding approximate solutions to (2.2) was devised by Stokes [Sto47] in the mid–1800's. It consists of the boundary perturbation philosophy we have termed "Field Expansions"—FE—(to distinguish it from the alternative "Operator Expansions" approach, see e.g. [NR03b, NR03c]) carried out to low (first or second) order. This method was expanded to higher orders by subsequent authors with the most recent attempts being those of Roberts [Rob83], Roberts & Schwartz [SR83], and Marchant & Roberts [MR87]. For ease of comparison with the results contained in these papers we adopt their notation in the current exposition of the FE approach.

For simplicity we consider (2.2) in the case of two dimensions ($d = 2$), 2π–periodicity, zero capillarity ($\sigma = 0$), and infinite depth ($h = \infty$). Following Roberts [Rob83] we define the surface velocities

$$U(x) = \partial_x \varphi(x,y)\big|_{y=\eta}, \qquad V(x) = \partial_y \varphi(x,y)\big|_{y=\eta},$$

and expand

$$\eta(x,\varepsilon) = \sum_{n\geq 1} \eta_n(x)\varepsilon^n, \quad \varphi(x,y,\varepsilon) = \sum_{n\geq 1} \varphi_n(x,y)\varepsilon^n, \quad c(\varepsilon) = \sum_{n\geq 0} c_n\varepsilon^n, \tag{2.10a}$$

$$U(x,\varepsilon) = \sum_{n\geq 1} U_n(x)\varepsilon^n, \quad V(x,\varepsilon) = \sum_{n\geq 1} V_n(x)\varepsilon^n. \tag{2.10b}$$

We find that we must solve

$$\begin{aligned}
&\Delta\varphi_n = 0 && y < 0 && (2.11a)\\
&\partial_y\varphi_n \to 0 && y \to -\infty && (2.11b)\\
&c_0 U_n + g\eta_n = Q_n - c_{n-1}U_1 && \text{at } y = 0 && (2.11c)\\
&-c_0\partial_x\eta_n + V_n = R_n + c_{n-1}\partial_x\eta_1 && \text{at } y = 0, && (2.11d)
\end{aligned}$$

where

$$\begin{aligned}
Q_n &= -\sum_{l=1}^{n-2} c_l U_{n-l} - \frac{1}{2}\sum_{l=1}^{n-1} U_l U_{n-l} - \frac{1}{2}\sum_{l=1}^{n-1} V_l V_{n-l}\\
R_n &= \sum_{l=1}^{n-2} c_l \partial_x\eta_{n-l} + \sum_{l=1}^{n-1} \partial_x\eta_l U_{n-l}.
\end{aligned}$$

To solve these equations we note that, on account of (2.11a), (2.11b), and the periodic boundary conditions, η_n and φ_n can be expressed as

$$\eta_n(x) = \sum_{k=-\infty}^{\infty} d_{n,k}e^{ikx}, \quad \varphi_n(x,y) = \sum_{k=-\infty}^{\infty} a_{n,k}e^{ikx+|k|y}. \tag{2.12}$$

To find forms for U_n and V_n we first write

$$e^{\gamma\eta} = \exp\left[\gamma\sum_{n\geq 1}\eta_n\varepsilon^n\right] = \sum_{n\geq 0} E_n(x;\gamma)\varepsilon^n.$$

As can be easily verified, the coefficients E_n are polynomials in γ of degree n which can be recursively found from the relations

$$E_0 = 1, \qquad E_n = \sum_{l=1}^{n} \frac{l}{n} \eta_l E_{n-l}(x;\gamma)\gamma. \tag{2.13}$$

Then, we have

$$\begin{aligned} U(x) &= \sum_{n\geq 1} \partial_x \ \varphi_n(x,y)\big|_{y=\eta} \varepsilon^n = \sum_{n\geq 1} \sum_{k=-\infty}^{\infty} a_{n,k}(ik)e^{ikx}e^{|k|\eta}\varepsilon^n \\ &= \sum_{n\geq 1} \sum_{k=-\infty}^{\infty} \left(\sum_{m\geq 0} E_m(x;|k|)\varepsilon^m \right) a_{n,k}(ik)e^{ikx}\varepsilon^n \\ &= \sum_{n\geq 1} \varepsilon^n \sum_{k=-\infty}^{\infty} (ik)a_{n,k}e^{ikx} + \sum_{n\geq 2} \varepsilon^n \sum_{l=1}^{n-1} \sum_{k=-\infty}^{\infty} E_{n-l}(x;|k|)(ik)a_{l,k}e^{ikx}, \end{aligned}$$

so that we can write $U_n = \bar{U}_n + \tilde{U}_n$ where

$$\bar{U}_n = \sum_{k=-\infty}^{\infty} (ik)a_{n,k}e^{ikx}, \quad \tilde{U}_n = \sum_{l=1}^{n-1} \sum_{k=-\infty}^{\infty} E_{n-l}(x;|k|)(ik)a_{l,k}e^{ikx}. \tag{2.14}$$

Similarly, $V_n = \bar{V}_n + \tilde{V}_n$ where

$$\bar{V}_n = \sum_{k=-\infty}^{\infty} |k|\, a_{n,k}e^{ikx}, \quad \tilde{V}_n = \sum_{l=1}^{n-1} \sum_{k=-\infty}^{\infty} E_{n-l}(x;|k|)\, |k|\, a_{l,k}e^{ikx}. \tag{2.15}$$

Finally, we can rewrite (2.11) as

$$\begin{aligned} &\Delta\varphi_n = 0 && y < 0 && (2.16a) \\ &\partial_y \varphi_n \to 0 && y \to -\infty && (2.16b) \\ &c_0\bar{U}_n + g\eta_n = Q_n - c_0\tilde{U}_n - c_{n-1}U_1 && \text{at } y = 0 && (2.16c) \\ &-c_0\partial_x\eta_n + \bar{V}_n = R_n - \tilde{V}_n + c_{n-1}\partial_x\eta_1 && \text{at } y = 0, && (2.16d) \end{aligned}$$

which, using (2.12), is equivalent to

$$\begin{pmatrix} ic_0 k & g \\ |k| & -ic_0 k \end{pmatrix} \begin{pmatrix} a_{n,k} \\ d_{n,k} \end{pmatrix} = \begin{pmatrix} S_{n,k} - (ic_{n-1}k)a_{1,k} \\ T_{n,k} + (ic_{n-1}k)d_{1,k} \end{pmatrix} \quad -\infty < k < \infty, \tag{2.17}$$

where

$$S_n = \sum_{k=-\infty}^{\infty} S_{n,k}e^{ikx}, \qquad T_n = \sum_{k=-\infty}^{\infty} T_{n,k}e^{ikx},$$

and

$$\begin{aligned} S_n &= -c_0\tilde{U}_n - \sum_{l=1}^{n-2} c_l U_{n-l} - \frac{1}{2}\sum_{l=1}^{n-1} U_l U_{n-l} - \frac{1}{2}\sum_{l=1}^{n-1} V_l V_{n-l} \\ T_n &= -\tilde{V}_n + \sum_{l=1}^{n-2} c_l \partial_x \eta_{n-l} + \sum_{l=1}^{n-1} \partial_x \eta_l U_{n-l}. \end{aligned}$$

The Field Expansions (FE) procedure consists of solving (2.17) recursively up to a specified order $n = N$ starting with relations of the form (2.6) where $k = \kappa_1$ and c_0 are chosen to satisfy $\Lambda_0(c_0, \kappa_1) = 0$. For example, choosing $\kappa_1 = c_0 = 1$ and normalizing $g = 1$ the linear part of the solution (taking α real) is, c.f. (2.6),

$$\eta_1(x) = 2\alpha \cos(x), \qquad \varphi_1(x, y) = -2\alpha e^y \sin(x). \tag{2.18}$$

The procedure can be carried out to arbitrarily high order N by recursively solving (2.17). Of course, by the choice of c_0, the matrix in (2.17) is singular at $k = \pm\kappa_1$ and a compatibility condition is required to ensure solvability. This is provided by an appropriate choice of c_{n-1} in (2.17) which closes the system of equations.

(*d*) *Cancellations*

It should be noted that this derivation of the FE recursions, (2.17), is purely formal in nature. Indeed, for instance, the recurrence entails spatial derivatives of the velocity potential of increasingly high order (c.f. (2.14), (2.15)) whose growth should be controlled if the series in (2.10) are to be shown to converge. On the other hand, if such control is to be based on the relations (2.17), it will demand, for instance, that we bound U_n, V_n recursively from (2.14), (2.15). Here, however, the only obvious bound will, by necessity, use the triangle inequality in the order l (c.f. (2.14), (2.15)). As we have shown in related applications of boundary perturbation approaches [NR01a, NR01b, NR03a, NR03b, NR03c] these bounds will consistently fail to provide useful growth control on any norm of the solutions, as they destroy significant cancellations that are present in the corresponding recurrences.

To substantiate this claim we next present a set of numerical experiments that demonstrate the existence of cancellations in (2.14), (2.15) as well as their implications in attempts at numerically simulating traveling water waves with high–order versions of the FE scheme. We begin by choosing specific parameters that give rise to a bifurcating solution: $d = 2$, $\sigma = 0$, $h = \infty$, $g = 1$, 2π–periodicity, $\kappa_1 = 1$, $c_0 = 1$, for which we present results of a suitable FE implementation up to order $N = 40$. In particular, all convolution products in S_n and T_n are performed using Fast Fourier acceleration in vectors of length $N_x = 128$ (which represent wavenumbers $[-N_x/2, N_x/2 - 1]$) greater than $2N = 80$ to prevent aliasing.

The first evidence we present concerns the accurate computation of the Fourier coefficients of the wave profile η. It is not difficult to see that, beginning with (2.18), $d_{n,n+p} = a_{n,n+p} = 0$ for all $p > 0$, while $d_{n,n}$ and $a_{n,n}$ will not be equal to zero and represent a "leading edge" of non–zero Fourier coefficients at order n. In Table 1 we report the results of the computation of $|d_{n,n}|$, via the FE recursions (2.17) with $\alpha = 1/2$ in (2.18), in both double and quadruple precision ($N_x = 128$). In the final column we treat the quadruple precision calculations as "exact solutions" and count the digits of accuracy in the double precision calculation. We point out the precipitous loss of accuracy in the coefficients $|d_{n,n}|$ through all orders of n, which rapidly accelerates beyond $n = 17$ resulting in approximations which contain *no accurate information* by $n = 27$.

In the previous calculation one could argue that for large n an accompanying factor of ε^n (where ε is typically much less than 1) in the approximation of $\eta(x)$ might disguise the inaccurate computation of $d_{n,n}$. However, as the next calculation illus-

Table 1. *Coefficients* $|d_{n,n}|$ *and digits of accuracy*

(Computation of the coefficients $|d_{n,n}|$, c.f. (2.12), in double and quadruple precision—only 16 digits are reported—and the digits of accuracy contained in the double precision calculation.)

n	double precision	quadruple precision	digits of accuracy
5	0.1627604166666668	0.1627604166666667	15
7	0.1823676215277779	0.1823676215277778	15
9	0.2316898890904013	0.2316898890904018	15
11	0.3172788927458329	0.3172788927458371	14
13	0.4567199344432450	0.4567199344432664	13
15	0.6812838291230653	0.6812838291231263	12
17	1.043768775207166	1.043768775207084	13
19	1.632677013377351	1.632677013390746	11
21	2.596641986288087	2.596641980321151	9
23	4.186275932441967	4.186277983801273	6
25	6.823624125009305	6.825974125230163	3
27	8.836865217279312	11.23738329006577	0

trates, such a hope is unfounded and the accurate computation of high–frequency information is *crucial* for a correct representation. To substantiate this claim, we next approximate a more physically relevant quantity, the L^2–norm of the wave form $\eta(x)$. More specifically, if we denote by

$$\eta_n^{N_x}(x;\varepsilon) = \sum_{j=0}^{n} \sum_{k=-N_x/2}^{N_x/2-1} d_{j,k} e^{ikx} \varepsilon^j, \tag{2.19}$$

which represents the FE approximation to $\eta(x)$, then in Figure 1 we present the difference (measured in L^2) between double precision and sextuple precision approximations of $\eta_n^{128}(x;0.3)$, and a highly resolved calculation (sextuple precision with $N_x = 128$ and $n = 40$); sextuple precision was necessary as we found that quadruple precision calculations were inadequate beyond $n = 33$. We note that at $\varepsilon = 0.3$ the sextuple precision calculation is fully converged at $n = 40$ indicating that $\varepsilon = 0.3$ is *within* the disc of convergence of the Taylor series (2.10). We point out in this figure the explosive divergence of the double precision calculation as n is increased. We also note the roughly linear shape of the curve on the log–linear axes indicating the exponential growth of errors.

3. Transformed Field Expansions

As the calculations of the previous section indicate, the cancellations in (2.17) are present for all n and increase in severity with increasing n. As explained above, this has consequences for more than just numerical simulation. Indeed, as we mentioned, the cancellations preclude the use of the most natural approach to estimating the convergence of the series (2.10) based on the derivation of bounds, e.g. of the form $\|\eta_n\|_{H^s} < CB^n$, from the recurrence (2.17).

However, as we explain next and further demonstrate in § 4 a direct estimation of the terms in the series (2.10) can be realized upon a change of independent variables in advance of the perturbation expansion, much as in the application of boundary

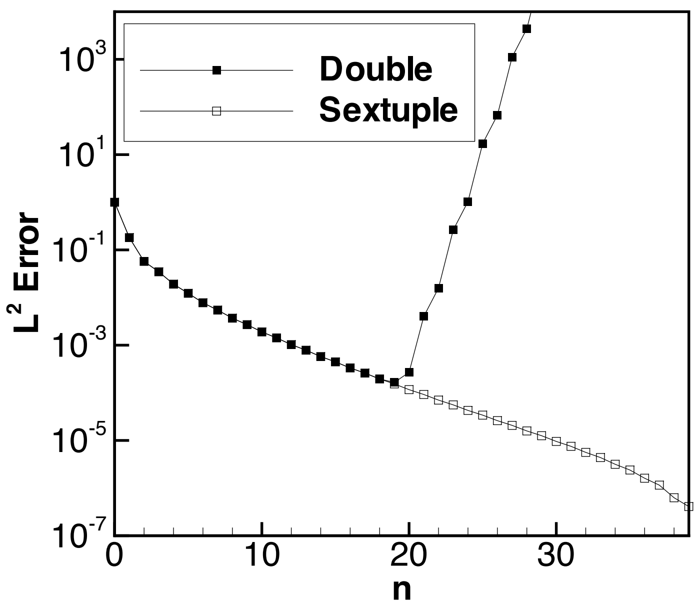

Figure 1. Comparison of double and sextuple precision computations of η_n^{128}, c.f. (2.19), with a highly resolved solution (sextuple precision calculation with $N_x = 128$, $n = 40$). Error is measured in the L^2 norm ($N_x = 128$, $0 \le n \le 39$, $\varepsilon = 0.3$, $\sigma = 0$, $h = \infty$, $g = 1$, $\kappa = 1$, $c_0 = 1$).

perturbation methods to boundary value problems [NR03b, NR03c]. Indeed, as in these latter applications, the transformation has the effect of implicitly accounting for all significant cancellations so that the terms in the corresponding recurrence can be inductively estimated. To derive these "Transformed Field Expansions" (TFE) we begin by considering the transformation

$$x' = x, \quad y' = \frac{y - \eta}{1 + \eta} \tag{3.1}$$

which maps the domain $S_{1,\eta}$ to the strip $S_{1,0}$. The equations (2.2) become, upon dropping primes,

$$\Delta\varphi(x, y) = F(x, y) \qquad \text{in } S_{1,0} \tag{3.2a}$$

$$\partial_y \varphi(x, -1) = \int_{P(\Gamma)} \partial_y \varphi(x, -1)\, dx, \quad \int_{P(\Gamma)} \varphi(x, -1)\, dx = 0 \tag{3.2b}$$

$$[c_0 \cdot \nabla_x]\, \varphi + [g - \sigma \Delta_x]\, \eta = Q(x) \qquad \text{at } y = 0 \tag{3.2c}$$

$$-\,[c_0 \cdot \nabla_x]\, \eta + \partial_y \varphi = R(x) \qquad \text{at } y = 0 \tag{3.2d}$$

where c_0 will be defined as in (2.7), $F(x, y) = \operatorname{div}_x \left[F^{(1)}(x, y)\right] + \partial_y F^{(2)}(x, y) +$

$F^{(3)}(x,y)$, and

$$F^{(1)}(x,y) = -\eta^2 \nabla_x \varphi - 2\eta \nabla_x \varphi + (1+y)(1+\eta)\nabla_x \eta \partial_y \varphi, \tag{3.3a}$$

$$F^{(2)}(x,y) = (1+y)(1+\eta)\nabla_x \eta \cdot \nabla_x \varphi - (1+y)^2 \left|\nabla_x \eta\right|^2 \partial_y \varphi, \tag{3.3b}$$

$$F^{(3)}(x,y) = (1+\eta)\nabla_x \eta \cdot \nabla_x \varphi - (1+y) \left|\nabla_x \eta\right|^2 \partial_y \varphi. \tag{3.3c}$$

The functions $Q(x)$ and $R(x)$ are represented by similar formulas.

To solve equations (3.2) in the transformed variables we now propose the following expansions for $\varepsilon \in \mathbf{R}^{d-1}$ and multi–index $n \in \mathbf{N}^{d-1}$

$$\varphi(x,y,\varepsilon) = \sum_{|n|\geq 1} \varphi_n(x,y)\varepsilon^n, \quad \eta(x,\varepsilon) = \sum_{|n|\geq 1} \eta_n(x)\varepsilon^n, \quad c(\varepsilon) = \sum_{|n|\geq 0} c_n \varepsilon^n \tag{3.4}$$

(the "Transformed Field Expansions") and find that we must solve the following problems

$$\Delta \varphi_n(x,y) = (1 - \delta_{|n|,0}) F_n(x,y) \qquad \text{in } S_{1,0} \tag{3.5a}$$

$$\partial_y \varphi_n(x,-1) = \int_{P(\Gamma)} \partial_y \varphi_n(x,-1)\, dx, \quad \int_{P(\Gamma)} \varphi_n(x,-1)\, dx = 0 \tag{3.5b}$$

$$\left[c_0 \cdot \nabla_x\right] \varphi_n(x,0) + \left[g - \sigma \Delta_x\right] \eta_n(x) + \sum_{\substack{j=1 \\ e_j \leq n}}^{d-1} \left[c_{n-e_j} \cdot \nabla_x\right] \varphi_{e_j}(x,0) = Q_n(x) \tag{3.5c}$$

$$-\left[c_0 \cdot \nabla_x\right] \eta_n(x) + \partial_y \varphi_n(x,0) - \sum_{\substack{j=1 \\ e_j \leq n}}^{d-1} \left[c_{n-e_j} \cdot \nabla_x\right] \eta_{e_j}(x,0) = R_n(x). \tag{3.5d}$$

Here $\delta_{k,p}$ is the Kronecker delta, $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ where e_j is non–zero at index j, and for multi–indices $m, n \in \mathbf{N}^{d-1}$, $m \leq n$ if $m_j \leq n_j$ for all $j = 1, \ldots, d-1$. Furthermore $F_n(x,y) = \mathrm{div}_x \left[F_n^{(1)}(x,y)\right] + \partial_y F_n^{(2)}(x,y) + F_n^{(3)}(x,y)$,

$$\begin{aligned} F_n^{(1)}(x,y) = &\sum_{|m|=2}^{|n|-1} \sum_{|l|=1}^{|m|-1} \eta_l \; \eta_{m-l} \; \nabla_x \varphi_{n-m} - 2 \sum_{|l|=1}^{|n|-1} \eta_l \; \nabla_x \varphi_{n-l} \\ &+ (1+y) \sum_{|m|=2}^{|n|-1} \sum_{|l|=1}^{|m|-1} \eta_l \; \nabla_x \eta_{m-l} \; \partial_y \varphi_{n-m} + (1+y) \sum_{|l|=1}^{|n|-1} \nabla_x \eta_l \; \partial_y \varphi_{n-l}, \end{aligned} \tag{3.6a}$$

$$\begin{aligned} F_n^{(2)}(x,y) = (1+y) &\sum_{|m|=2}^{|n|-1} \sum_{|l|=1}^{|m|-1} \eta_l \; \nabla_x \eta_{m-l} \cdot \nabla_x \varphi_{n-m} + (1+y) \sum_{|l|=1}^{|n|-1} \nabla_x \eta_l \cdot \nabla_x \varphi_{n-l} \\ &- (1+y)^2 \sum_{|m|=2}^{|n|-1} \sum_{|l|=1}^{|m|-1} \nabla_x \eta_l \cdot \nabla_x \eta_{m-l} \; \partial_y \varphi_{n-m}, \end{aligned} \tag{3.6b}$$

$$F_n^{(3)}(x,y) = \sum_{|m|=2}^{|n|-1} \sum_{|l|=1}^{|m|-1} \eta_l \; \nabla_x \eta_{m-l} \cdot \nabla_x \varphi_{n-m} + \sum_{|l|=1}^{|n|-1} \nabla_x \eta_l \cdot \nabla_x \varphi_{n-l}$$
$$-(1+y) \sum_{|m|=2}^{|n|-1} \sum_{|l|=1}^{|m|-1} \nabla_x \eta_l \cdot \nabla_x \eta_{m-l} \; \partial_y \varphi_{n-m}. \tag{3.6c}$$

The functions Q_n and R_n can be similarly derived. Note that $F_n^{(l)}$ (and Q_n, and R_n) depend only on $\{\eta_j\}_{|j|<|n|}$, $\{\varphi_j\}_{|j|<|n|}$, and $\{c_j\}_{|j|<|n|-1}$, and vanish for $|n| = 1$.

4. Analyticity of Solutions

Clearly, the nature of (3.5) is quite similar to that of (2.16); however, there are some important differences. Most notably, in contrast to (2.16), the right–hand sides of (3.5) contain only derivatives of order one or two which act either on the field, φ_l, or the surface shape, η_l. In what follows we show that this, in fact, allows for the inductive establishment of bounds:

$$\|\varphi_n\|_X \le CB^{|n|} \quad \|\eta_n\|_Y \le CB^{|n|} \quad |c_n| \le CB^{|n|}$$

in appropriate function spaces X and Y, and for some constants $C, B \ge 0$. Furthermore, we will show that *all* spatial derivatives of φ_n and η_n can be similarly bounded implying that all quantities are *jointly* analytic with respect to all arguments.

To set notation we recall that any L^2 function f periodic on a $(d-1)$ dimensional lattice $\Gamma \subset \mathbf{R}^{d-1}$ can be represented as

$$f(x) = \sum_{k \in \Gamma'} \hat{f}(k) e^{ik\cdot x} \tag{4.1}$$

where Γ' is the conjugate lattice to Γ. Additionally, if an L^2 function $u(x,y)$ is periodic in x with respect to Γ and square integrable in the y variable on $[-1,0]$ then

$$u(x,y) = \sum_{k \in \Gamma'} \hat{u}(k,y) e^{ik\cdot x}. \tag{4.2}$$

Using this representation we can define the L^2 based Sobolev spaces

$$H^s = \{u \in L^2 \mid \|u\|_{H^s} < \infty\} \tag{4.3}$$

for $s \in \mathbf{Z}$, and where

$$\|u(x,y)\|_{H^s}^2 \equiv \sum_{j=0}^{s} \sum_{k \in \Gamma'} \langle k \rangle^{2s-2j} \int_{-1}^{0} \left| \partial_y^j \hat{u}(k,y) \right|^2 \, dy \tag{4.4}$$

and $\langle k \rangle = \sqrt{1+k^2}$. Note that if $f = f(x)$ depends on x alone then the space H^s for $s \in \mathbf{R}$ can be defined by the norm

$$\|f(x)\|_{H^s}^2 \equiv \sum_{k \in \Gamma'} \langle k \rangle^{2s} \left| \hat{f}(k) \right|^2. \tag{4.5}$$

For future reference we note the following algebra property for H^s [Ada75].

Lemma 4.1. *If $s > d/2$ then H^s is an algebra, i.e. for $u, v \in H^s$*

$$\|uv\|_{H^s} \leq M\,\|u\|_{H^s}\,\|v\|_{H^s} \tag{4.6}$$

for a constant $M = M(d,s)$ depending only on d and s.

In the case of non-zero surface tension ($\sigma > 0$), our main result is:

Theorem 4.2. *Given an integer $s > d/2$, if $|n| \geq 1$ the solutions $\varphi_n(x,y)$, $\eta_n(x)$, and c_{n-e_j} of (3.5) ($\sigma > 0$) satisfy*

$$\left\|\frac{\partial_x^k \partial_y^l}{(|k|+l)!}\varphi_n\right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p}\frac{D^l}{(l+1)^2}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2} \tag{4.7a}$$

$$\left\|\frac{\partial_x^k}{|k|!}\eta_n\right\|_{H^{s+5/2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}, \quad |c_{n-e_j}| \leq C_1\frac{B^{|n|-1}}{|n|^p} \tag{4.7b}$$

for all $j = 1, \ldots, d-1$, $p > d-1$, and some constants $C_1, B, D, A > 0$.

We will prove Theorem 4.2 by induction on l; thus our first objective is to establish the result for $l = 0$. This is done in the following Lemma, for all k, with an induction on the order $|n|$.

Lemma 4.3. *Given an integer $s > d/2$, if $|n| \geq 1$ the solutions $\varphi_n(x,y)$, $\eta_n(x)$, and c_{n-e_j} of (3.5) ($\sigma > 0$) satisfy*

$$\left\|\frac{\partial_x^k}{|k|!}\varphi_n\right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}$$

$$\left\|\frac{\partial_x^k}{|k|!}\eta_n\right\|_{H^{s+5/2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}, \quad |c_{n-e_j}| \leq C_1\frac{B^{|n|-1}}{|n|^p}$$

for all $j = 1, \ldots, d-1$, $p > d-1$, and some constants $C_1, B, A > 0$.

To prove Lemma 4.3 we need two Lemmas: The following which estimates the right hand sides (3.6) of the inhomogeneous problems, and the sequel which provides estimates on solutions of these problems.

Lemma 4.4. *Given an integer $s > d/2$, suppose that*

$$\left\|\frac{\partial_x^k}{|k|!}\varphi_n\right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2} \quad \forall k$$

$$\left\|\frac{\partial_x^k}{|k|!}\eta_n\right\|_{H^{s+5/2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2} \quad \forall k, \quad |c_{n-e_j}| \leq C_1\frac{B^{|n|-1}}{|n|^p},$$

for all $1 \leq n < N$ (i.e. $n_j < N_j$), $p > d-1$, and for some constants $C_1, B, A > 0$. Then there exists a constant C_2 such that the functions $F_N^{(j)}$, $Q_N^{(j)}$, and R_N in (3.6) *satisfy*

$$\left\| \frac{\partial_x^k}{|k|!} F_N^{(j)} \right\|_{H^s} \leq C_1 C_2 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \quad \forall k \tag{4.8a}$$

$$\left\| \frac{\partial_x^k}{|k|!} Q_N^{(j)} \right\|_{H^{s+1/2}} \leq C_1 C_2 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \quad \forall k \tag{4.8b}$$

$$\left\| \frac{\partial_x^k}{|k|!} R_N \right\|_{H^{s+1/2}} \leq C_1 C_2 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \quad \forall k. \tag{4.8c}$$

Proof. (Lemma 4.4)
For the sake of brevity consider the first term of $F_N^{(3)}$,

$$Z_1 = \sum_{|m|=2}^{|n|-1} \sum_{|l|=1}^{|m|-1} \eta_l \ \eta_{m-l} \ \nabla_x \varphi_{n-m};$$

every other term in F_N, Q_N, and R_N can be similarly estimated. We begin

$$\left\| \frac{\partial_x^k}{|k|!} Z_1 \right\|_{H^s} \leq \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} M^2 \sum_{\sigma \leq \tau} \sum_{\tau \leq k} \frac{k! \, |\sigma|! \, |\tau-\sigma|! \, |k-\tau|!}{|k|! \sigma! (\tau-\sigma)! (k-\tau)!} \left\| \frac{\partial_x^\sigma}{|\sigma|!} \eta_l \right\|_{H^s}$$
$$\times \left\| \frac{\partial_x^{\tau-\sigma}}{|\tau-\sigma|!} \nabla_x \eta_{m-l} \right\|_{H^s} \left\| \frac{\partial_x^{k-\tau}}{|k-\tau|!} \nabla_x \varphi_{N-m} \right\|_{H^s}.$$

Since

$$\frac{k! \, |\sigma|! \, |\tau-\sigma|! \, |k-\tau|!}{|k|! \sigma! (\tau-\sigma)! (k-\tau)!} \leq 1$$

we can deduce that

$$\left\| \frac{\partial_x^k}{|k|!} Z_1 \right\|_{H^s} \leq \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} M^2 \sum_{\sigma \leq \tau} \sum_{\tau \leq k} C_1 \frac{B^{|l|-1}}{|l|^p} \prod_{q=1}^{d-1} \frac{A^{\sigma_q}}{(\sigma_q+1)^2}$$
$$\times C_1 \frac{B^{|m-l|-1}}{|m-l|^p} \prod_{q=1}^{d-1} \frac{A^{\tau_q-\sigma_q}}{(\tau_q-\sigma_q+1)^2} C_1 \frac{B^{|N-m|-1}}{|N-m|^p} \prod_{q=1}^{d-1} \frac{A^{k_q-\tau_q}}{(k_q-\tau_q+1)^2}.$$

Continuing,

$$\left\| \frac{\partial_x^k}{|k|!} Z_1 \right\|_{H^s} \leq M^2 C_1^3 \frac{B^{|N|-3}}{|N|^p} S^{2(d-1)} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} \frac{|N|^p}{|l|^p \, |m-l|^p \, |N-m|^p},$$

and

$$\left\| \frac{\partial_x^k}{|k|!} Z_1 \right\|_{H^s} \leq \left[M^2 C_1^3 \frac{S^{2(d-1)} \Sigma_{d-1}^2}{B} \right] \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$

where

$$S = \max_k \sum_{\tau=0}^{k} \frac{k^2}{(\tau+1)^2 (k-\tau-1)^2}, \quad \Sigma_{d-1} \equiv \max_m \sum_{|l|=1}^{|m|-1} \frac{|m|^p}{|l|^p \, |m-l|^p},$$

which is bounded uniformly in $|m|$ for $p > d-1$. The proof is complete provided $B > MC_1 S^{d-1}\Sigma_{d-1}$. □

The second Lemma necessary to prove Lemma 4.3 is now presented; the proof, based on classical elliptic estimates, is given in Appendix A.

Lemma 4.5. *Consider any integer $s \geq 0$. Given linearly independent wavenumbers $\kappa_1, \dots \kappa_{d-1} \in \Gamma' \subset \mathbf{R}^{d-1}$ there exists a unique speed $c = (c_1, \dots, c_{d-1}) \in \mathbf{R}^{d-1}$ satisfying (2.8) such that $R_j > 0$. Given this c, if for all multi–indices $|n| = \bar{n}$, $p_n \in H^s$, $q_n \in H^{s+1/2}$, and $r_n \in H^{s+1/2}$ then there exist for all $|n| = \bar{n}$ real solutions $w_n \in H^{s+2}$, $v_n \in H^{s+5/2}$, and μ_{n-e_j} $(j = 1, \dots, d)$ of*

$$\Delta w_n(x,y) = p_n(x,y) \qquad \text{in } S_{1,0} \tag{4.9a}$$

$$\partial_y w_n(x,-1) = \int_{P(\Gamma)} \partial_y w_n(x,-1)\, dx, \quad \int_{P(\Gamma)} w_n(x,-1)\, dx = 0 \tag{4.9b}$$

$$(g - \sigma\Delta_x)v_n(x) + [c\cdot\nabla_x]\, w_n(x,0) + \sum_{\substack{j=1\\ e_j\leq n}}^{d-1} \left[\mu_{n-e_j}\cdot\nabla_x\right] b_{e_j}(x) = q_n(x) \tag{4.9c}$$

$$-[c\cdot\nabla_x]\, v_n(x) + \partial_y w_n(x,0) - \sum_{\substack{j=1\\ e_j\leq n}}^{d-1} \left[\mu_{n-e_j}\cdot\nabla_x\right] f_{e_j}(x) = r_n(x) \tag{4.9d}$$

where

$$b_{e_j}(x) = \alpha_j i(g + \sigma|\kappa_j|^2)e^{i\kappa_j\cdot x} - \bar{\alpha}_j i(g+\sigma|\kappa_j|^2)e^{-i\kappa_j\cdot x} \tag{4.10a}$$

$$f_{e_j}(x) = \alpha_j(c\cdot\kappa_j)e^{i\kappa_j\cdot x} + \bar{\alpha}_j(c\cdot\kappa_j)e^{-i\kappa_j\cdot x}. \tag{4.10b}$$

If, in addition, we require that

$$\int_{P(\Gamma)} v_n(x)e^{\pm i\kappa_j\cdot x}\, dx = 0 \tag{4.11}$$

then this solution is unique. Furthermore there exists a constant C_e such that the solutions satisfy

$$\|w_n\|_{H^{s+2}} \leq C_e\left[\|p_n\|_{H^s} + \|q_n\|_{H^{s-1/2}} + \|r_n\|_{H^{s+1/2}}\right] \tag{4.12a}$$

$$\|v_n\|_{H^{s+5/2}} \leq C_e\left[\|p_n\|_{H^s} + \|q_n\|_{H^{s+1/2}} + \|r_n\|_{H^{s+1/2}}\right] \tag{4.12b}$$

$$\left|\mu_{n-e_j}\right| \leq C_e\left[\|p_n\|_{H^s} + \|q_n\|_{H^{s+1/2}} + \|r_n\|_{H^{s+1/2}}\right]. \tag{4.12c}$$

We can now complete the proof of Lemma 4.3.

Proof. (Lemma 4.3)
The proof proceeds via induction on $|n|$; since φ_n, η_n, c_n satisfy (3.5), $\frac{\partial_x^k}{|k|!}\varphi_n$, $\frac{\partial_x^k}{|k|!}\eta_n$,

and c_{n-1} satisfy

$$\Delta \frac{\partial_x^k}{|k|!}\varphi_n = (1-\delta_{n,1})\frac{\partial_x^k}{|k|!}F_n \qquad \text{in } S_{1,0} \tag{4.13a}$$

$$\partial_y \frac{\partial_x^k}{|k|!}\varphi_n(x,-1) = \int_{P(\Gamma)} \partial_y \frac{\partial_x^k}{|k|!}\varphi_n(x,-1)\;dx, \quad \int_{P(\Gamma)} \frac{\partial_x^k}{|k|!}\varphi_n(x,-1)\;dx = 0 \tag{4.13b}$$

$$[c_0\cdot\nabla_x]\,\frac{\partial_x^k}{|k|!}\varphi_n(x,0) + [g-\sigma\Delta_x]\,\frac{\partial_x^k}{|k|!}\eta_n(x) + (1-\delta_{n,1})\,[c_{n-1}\cdot\nabla_x]\,\frac{\partial_x^k}{|k|!}\varphi_1(x,0)$$
$$= (1-\delta_{n,1})\frac{\partial_x^k}{|k|!}Q_n(x) \tag{4.13c}$$

$$-[c_0\cdot\nabla_x]\,\frac{\partial_x^k}{|k|!}\eta_n(x,0) + \partial_y\frac{\partial_x^k}{|k|!}\varphi_n(x,0) - (1-\delta_{n,1})\,[c_{n-1}\cdot\nabla_x]\,\frac{\partial_x^k}{|k|!}\eta_1(x,0)$$
$$= (1-\delta_{n,1})\frac{\partial_x^k}{|k|!}R_n(x). \tag{4.13d}$$

In the case $|n|=1$ we have the *explicit* formulas

$$\varphi_{e_j}(x,y) = \alpha_{k_j} i(g+\sigma\,|k_j|^2)\frac{\cosh(|k_j|\,(y+1))}{\cosh(|k_j|)}e^{ik_j\cdot x}$$
$$-\bar{\alpha}_{k_j} i(g+\sigma\,|k_j|^2)\frac{\cosh(|k_j|\,(y+1))}{\cosh(|k_j|)}e^{-ik_j\cdot x} \tag{4.14a}$$
$$\eta_{e_j}(x) = \alpha_{k_j}(c\cdot k_j)e^{ik_j\cdot x} + \bar{\alpha}_{k_j}(c\cdot k_j)e^{-ik_j\cdot x} \tag{4.14b}$$

and, for $c_0\in\mathbf{R}^{d-1}$, the set of $(d-1)$ equations $\Lambda_\sigma(c_0,\kappa_j)=0$. From these formulas the estimates for $|n|=1$ follow easily. Next, consider order $|N|$ and note that $N=\nu+e_j$ for some multi–index ν. Suppose that

$$\left\|\frac{\partial_x^k}{|k|!}\varphi_n\right\|_{H^{s+2}} \le C_1\frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}$$
$$\left\|\frac{\partial_x^k}{|k|!}\eta_n\right\|_{H^{s+5/2}} \le C_1\frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}, \quad |c_{n-e_j}| \le C_1\frac{B^{|n|-1}}{|n|^p}$$

for all k and for all $1\le|n|<|N|$, $n<N$. Using Lemma 4.5 we have

$$\left\|\frac{\partial_x^k}{|k|!}\varphi_{\nu+e_j}\right\|_{H^{s+2}} \le C_e\left[\left\|\frac{\partial_x^k}{|k|!}F_{\nu+e_j}\right\|_{H^s} + \left\|\frac{\partial_x^k}{|k|!}Q_{\nu+e_j}\right\|_{H^{s-1/2}} + \left\|\frac{\partial_x^k}{|k|!}R_{\nu+e_j}\right\|_{H^{s+1/2}}\right]$$
$$\left\|\frac{\partial_x^k}{|k|!}\eta_{\nu+e_j}\right\|_{H^{s+5/2}} \le C_e\left[\left\|\frac{\partial_x^k}{|k|!}F_{\nu+e_j}\right\|_{H^s} + \left\|\frac{\partial_x^k}{|k|!}Q_{\nu+e_j}\right\|_{H^{s+1/2}} + \left\|\frac{\partial_x^k}{|k|!}R_{\nu+e_j}\right\|_{H^{s+1/2}}\right]$$
$$|c_\nu| \le C_e\left[\left\|\frac{\partial_x^k}{|k|!}F_{\nu+e_j}\right\|_{H^s} + \left\|\frac{\partial_x^k}{|k|!}Q_{\nu+e_j}\right\|_{H^{s+1/2}} + \left\|\frac{\partial_x^k}{|k|!}R_{\nu+e_j}\right\|_{H^{s+1/2}}\right].$$

Finally, using Lemma 4.4 we obtain

$$\left\|\frac{\partial_x^k}{|k|!}\varphi_{\nu+e_j}\right\|_{H^{s+2}} \le C_e 6 C_1 C_2 \frac{B^{|N|-2}}{|N|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}$$

$$\left\|\frac{\partial_x^k}{|k|!}\eta_{\nu+e_j}\right\|_{H^{s+5/2}} \le C_e 6 C_1 C_2 \frac{B^{|N|-2}}{|N|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}$$

$$|c_\nu| \le C_e 6 C_1 C_2 \frac{B^{|N|-2}}{|N|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}.$$

The proof is complete provided that $B > 6C_eC_2$ and $p > d-1$. □

Lemma 4.3 proves (4.7) for $l = 0$. In order to complete the induction for $l > 0$ we need the following two results.

Lemma 4.6. *Given an integer $s > 0$ the following estimate holds*

$$\left\|\frac{\partial_x^k\partial_y^L}{(|k|+L)!}\varphi_{e_j}\right\|_{H^{s+2}} \le C_1\frac{D^L}{(L+1)^2}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}\quad \forall k, L \tag{4.15}$$

for some constants $C_1, D, A > 0$.

Proof. (Lemma 4.6)
The proof comes immediately from the explicit formula (4.14) for φ_{e_j}. □

Lemma 4.7. *Given an integer $s > d/2$, suppose that*

$$\left\|\frac{\partial_x^k\partial_y^l}{(|k|+l)!}\varphi_n\right\|_{H^{s+2}} \le C_1\frac{B^{|n|-1}}{|n|^p}\frac{D^l}{(l+1)^2}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2} \tag{4.16a}$$

$$\left\|\frac{\partial_x^k}{|k|!}\eta_n\right\|_{H^{s+5/2}} \le C_1\frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2},\quad \left|c_{n-e_j}\right| \le C_1\frac{B^{|n|-1}}{|n|^p} \tag{4.16b}$$

for all indices $k \ge 0$, $|n| \ge 1$, when $l < L$; for all $k \ge 0$, $|n| < |N|$ when $l = L$; and $p > d-1$. Then there exists a constant C_3 such that F_N, Q_N, and R_N in (3.6) *satisfy*

$$\left\|\frac{\partial_x^k\partial_y^{L-1}}{(|k|+L)!}F_N\right\|_{H^{s+1}} \le C_1C_3\frac{B^{|N|-2}}{|N|^p}\frac{D^L}{(L+1)^2}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}\quad \forall k \tag{4.17a}$$

$$\left\|\frac{\partial_x^k}{|k|!}Q_N\right\|_{H^{s+1/2}} \le C_1C_3\frac{B^{|N|-2}}{|N|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}\quad \forall k \tag{4.17b}$$

$$\left\|\frac{\partial_x^k}{|k|!}R_N\right\|_{H^{s+1/2}} \le C_1C_3\frac{B^{|N|-2}}{|N|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}\quad \forall k. \tag{4.17c}$$

Proof. (Lemma 4.7)
Estimates (4.17b) & (4.17c) are true by Lemma 4.4. For brevity, in regard to (4.17a) let us consider only the third portion of $F_N^{(2)}$, namely

$$Z_3(x,y) \equiv (1+y)^2 \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} \nabla_x \eta_l \cdot \nabla_x \eta_{m-l} \partial_y \varphi_{N-m}.$$

Then

$$\begin{aligned}\left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \partial_y Z_3 \right\|_{H^{s+1}} \leq & \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} M^2 \sum_{\sigma \leq \tau} \sum_{\tau \leq k} \frac{k!\,|\sigma|!\,|\tau-\sigma|!(|k-\tau|+L)!}{\sigma!(\tau-\sigma)!(k-\tau)!(|k|+L)!} \\ & \times \left\| \frac{\partial_x^\sigma}{|\sigma|!} \nabla_x \eta_l \right\|_{H^{s+1}} \left\| \frac{\partial_x^{\tau-\sigma}}{|\tau-\sigma|!} \nabla_x \eta_{m-l} \right\|_{H^{s+1}} \\ & \times \left\| \frac{\partial_x^{k-\tau} \partial_y^L}{(|k-\tau|+L)!} \left[(1+y)^2 \partial_y \varphi_{N-m}\right] \right\|_{H^{s+1}}.\end{aligned}$$

Since

$$\frac{k!\,|\sigma|!\,|\tau-\sigma|!(|k-\tau|+L)!}{\sigma!(\tau-\sigma)!(k-\tau)!(|k|+L)!} \leq 1$$

and

$$\begin{aligned}\partial_y^L \left[(1+y)^2 \partial_y \varphi_{N-m}\right] = & (1+y)^2 \partial_y^{L+1} \varphi_{N-m} \\ & + 2L(1+y) \partial_y^L \varphi_{N-m} \\ & + L(L-1) \partial_y^{L-1} \varphi_{N-m}\end{aligned}$$

we can continue

$$\begin{aligned}\left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \partial_y Z_3 \right\|_{H^{s+1}} \leq & \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} M^2 \sum_{\sigma \leq \tau} \sum_{\tau \leq k} C_1 \frac{B^{|l|-1}}{|l|^p} \prod_{q=1}^{d-1} \frac{A^{\sigma_q}}{(\sigma_q+1)^2} \\ & \times C_1 \frac{B^{|m-l|-1}}{|m-l|^p} \prod_{q=1}^{d-1} \frac{A^{\tau_q-\sigma_q}}{(\tau_q-\sigma_q+1)^2} \\ & \times \left\{ Y^2 M^2 \left\| \frac{\partial_x^{k-\tau} \partial_y^L}{(|k-\tau|+L)!} \varphi_{N-m} \right\|_{H^{s+2}} \right. \\ & + 2LYM \left\| \frac{\partial_x^{k-\tau} \partial_y^{L-1}}{(|k-\tau|+L)!} \varphi_{N-m} \right\|_{H^{s+2}} \\ & \left. + L(L-1) \left\| \frac{\partial_x^{k-\tau} \partial_y^{L-2}}{(|k-\tau|+L)!} \varphi_{N-m} \right\|_{H^{s+2}} \right\}.\end{aligned}$$

Using (4.16) we obtain

$$\left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \partial_y Z_3 \right\|_{H^s} \leq C_1^2 M^2 \frac{B^{|N|-3}}{|N|^p} \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} \frac{|N|^p}{|l|^p\,|m-l|^p\,|N-m|^p}$$
$$\times \prod_{j=1}^{d-1} \left\{ \frac{A^{k_j}}{(k_j+1)^2} \sum_{\sigma \leq \tau} \sum_{\tau \leq k} \frac{(k_j+1)^2}{(\sigma_j+1)^2 (\tau_j - \sigma_j + 1)^2 (k_j - \tau_j + 1)^2} \right\}$$
$$\times \left\{ Y^2 M^2 \frac{D^L}{(L+1)^2} + \frac{2LYM}{(|k-\tau|+L)} \frac{D^{L-1}}{L^2} + \frac{L(L-1)}{(|k-\tau|+L)(|k-\tau|+L-1)} \frac{D^{L-2}}{(L-1)^2} \right\},$$

and,

$$\left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \partial_y Z_3 \right\|_{H^s} \leq C_1 \frac{C_1 M^2 S^{2(d-1)}}{B} \left\{ Y^2 M^2 + \frac{2LYM(L+1)^2}{(|k-\tau|+L)L^2 D} + \frac{L(L-1)(L+1)^2}{(|k-\tau|+L)(|k-\tau|+L-1)(L-1)^2 D^2} \right\}$$
$$\times \frac{B^{|N|-2}}{|N|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$
$$\leq C_1 C_3 \frac{B^{|N|-2}}{|N|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$

provided that C_3 is chosen appropriately. □

We are now in a position to prove Theorem 4.2.

Proof. (Theorem 4.2)
We begin by noting that Lemma 4.3 has already established (4.7b) so we need only focus on (4.7a). For this estimate we work by induction on l. For $l = 0$ and any $k \geq 0$, $|n| \geq 1$ we use Lemma 4.3. Now we assume

$$\left\| \frac{\partial_x^k \partial_y^l}{(|k|+l)!} \varphi_n \right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \frac{D^l}{(l+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$

for all $l < L$ and any $k \geq 0$ and $|n| \geq 1$, and seek to prove

$$\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_n \right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$

for any $k \geq 0$ and $|n| \geq 1$. We accomplish this via a second induction on $|n|$. Lemma 4.6 establishes (4.7a) in the case $|n| = 1$. We now assume that

$$\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_n \right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$

for all $|n| < |N|$ and seek to prove

$$\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} \leq C_1 \frac{B^{|N|-1}}{|N|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}.$$

We make the estimate

$$\begin{aligned}
\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} &\leq \left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_N \right\|_{H^{s+1}} + \left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \nabla_x \varphi_N \right\|_{H^{s+1}} \\
&\quad + \left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \partial_y \varphi_N \right\|_{H^{s+1}} \\
&\leq \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \nabla_x \varphi_N \right\|_{H^{s+2}} \\
&\quad + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \Delta_x \varphi_N \right\|_{H^{s+1}} + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} F_N \right\|_{H^{s+1}} \\
&\leq \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \nabla_x \varphi_N \right\|_{H^{s+2}} \\
&\quad + C(d) \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \nabla_x \varphi_N \right\|_{H^{s+2}} + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} F_N \right\|_{H^{s+1}}
\end{aligned}$$

where C is a generic function of dimension alone and we have used the fact that φ_N solves (3.5). Finally, using Lemma 4.7 we have

$$\begin{aligned}
\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} &\leq C_1 \left[1 + C(d)A\right] \frac{B^{|N|-1}}{|N|^p} \frac{D^{L-1}}{L^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \\
&\quad + C_1 C_3 \frac{B^{|N|-2}}{|N|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}
\end{aligned}$$

which completes the proof provided that $D > (1 + C(d)A)$ and $B > C_3$. □

We remark here that the same existence and analyticity proofs given above for the case of capillary–gravity waves ($\sigma > 0$) can also be given for pure gravity waves ($\sigma = 0$) provided that $d = 2$. The key differences lie in the different nature of Λ_σ when $\sigma = 0$, and the elliptic estimate, Lemma 4.5.

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Appendix A. Elliptic Estimate

First, we state two lemmas which are needed in the proof of Theorem 4.5.

Lemma 4.8. *If the unknowns* $\mu_m \in \mathbf{R}^{d-1}$, $m \in \mathbf{N}^{d-1}$, $|m| = \bar{n} - 1 \geq 0$ *are constrained by the linear equations*

$$E_{n,j} : \kappa_j \cdot \mu_{n-e_j} = R_{n,j}, \quad \forall \ |n| = \bar{n} \tag{A 1}$$

and all $j = 1, \ldots, d-1$ *such that* $n \geq e_j$, *where* $\kappa_1, \ldots, \kappa_{d-1}$ *are linearly independent, then a unique solution of* (A 1) *exists.*

Lemma 4.9. *The convolution integrals*

$$T_1(F)(y) = \int_{-1}^{y} e^{|k|(s-y)} F(s)\, ds, \quad T_2(F)(y) = \int_{y}^{0} e^{|k|(y-s)} F(s)\, ds,$$

($k \neq 0$) satisfy the estimates

$$\|T_1(F)\|_{L^2} \leq \frac{1}{|k|} \|F\|_{L^2}, \quad \|T_2(F)\|_{L^2} \leq \frac{1}{|k|} \|F\|_{L^2}$$
$$|T_1(F)(0)| \leq \frac{1}{\sqrt{2|k|}} \|F\|_{L^2}, \quad |T_2(F)(-1)| \leq \frac{1}{\sqrt{2|k|}} \|F\|_{L^2}.$$

We will now establish the elliptic estimate Lemma 4.5.

Proof. (Lemma 4.5)
The periodicity of solutions of (4.9) permit the Fourier series expansions

$$w_n(x,y) = \sum_{k\in\Gamma'} \hat{w}_n(k,y) e^{ik\cdot x}, \quad v_n(x,y) = \sum_{k\in\Gamma'} \hat{v}_n(k) e^{ik\cdot x}.$$

If $k \neq 0$ variation of parameters gives a solution of (4.9a)

$$\hat{w}_n(k,y) = A_n(k) e^{|k|y} - \frac{1}{2|k|} T_2(y) + B_n(k) e^{-|k|y} - \frac{1}{2|k|} T_1(y), \tag{A 2}$$

where

$$T_1(y) \equiv \int_{-1}^{y} e^{|k|(s-y)} \hat{p}_n(k,s)\, ds, \quad T_2(y) \equiv \int_{y}^{0} e^{|k|(y-s)} \hat{p}_n(k,s)\, ds.$$

Then (4.9b)–(4.9d) can be used to solve for $A_n(k)$, $B_n(k)$, and $\hat{v}_n(k)$. Provided that $k \neq 0$ and $k \neq \pm\kappa_j$ the solution of this problem is

$$\begin{aligned}
\hat{w}_n(k,y) &= \frac{1}{\Lambda_\sigma(c,k)\cosh(|k|)}\left\{-(ic\cdot k)\hat{q}_n(k)\cosh(|k|(y+1))\right.\\
&\quad -(g+\sigma|k|^2)\hat{r}_n(k)\cosh(|k|(y+1)) + \frac{(c\cdot k)^2T_1(0)}{2|k|}\cosh(|k|(y+1))\\
&\quad + \frac{(g+\sigma|k|^2)|k|T_1(0)}{2|k|}\cosh(|k|(y+1)) + \frac{(c\cdot k)^2T_2(-1)}{2|k|}\sinh(|k|y)\\
&\quad \left.+\frac{(g+\sigma|k|^2)|k|T_2(-1)}{2|k|}\cosh(|k|y)\right\} - \frac{1}{2|k|}T_2(y) - \frac{1}{2|k|}T_1(y)\\
\hat{v}_n(k) &= \frac{-(ic\cdot k)}{2\Lambda_\sigma(c,k)\cosh(|k|)}\left\{e^{|k|}T_1(0)+T_2(-1)\right\} - \frac{|k|\tanh(|k|)}{\Lambda_\sigma(c,k)}\hat{q}_n(k) + \frac{(ic\cdot k)}{\Lambda_\sigma(c,k)}\hat{r}_n(k).
\end{aligned}$$

In the case $k=0$, variation of parameters and (4.9) require that

$$\begin{aligned}
\hat{w}_n(0,y) &= (y+1)\hat{r}_n(0) - y\int_y^0 \hat{p}_n(0,s)\,ds - \int_{-1}^y s\hat{p}_n(0,s)\,ds - \int_{-1}^0 \hat{p}_n(0,s)\,ds\\
\hat{v}_n(0) &= \frac{\hat{q}_n(0)}{g}.
\end{aligned}$$

where we have used (4.9b) to uniquely specify $\hat{w}_n(0,y)$.

For $k=\kappa_j$ (and similarly for $k=-\kappa_j$) we again use the variation of parameters formula (A 2) and seek $A_n(\kappa_j)$, $B_n(\kappa_j)$, and $\hat{v}_n(\kappa_j)$. Combining (4.9c) & (4.9d) it can be shown that

$$\begin{aligned}
&-(c\cdot\kappa_j)^2\hat{w}_j(\kappa_j,0) + (g+\sigma|\kappa_j|^2)\partial_y\hat{w}_j(\kappa_j,0) =\\
&\qquad 2i(c\cdot\kappa_j)(\mu_n\cdot\kappa_j)(g+\sigma|\kappa_j|^2)\alpha_j + (ic\cdot\kappa_j)\hat{q}_j(\kappa_j) + (g+\sigma|\kappa_j|^2)\hat{r}_j(\kappa_j) \qquad (A\,3)
\end{aligned}$$

must hold. Equations (4.9b) & (A 3) result in a linear system of equations for $A_n(\kappa_j)$ and $B_n(\kappa_j)$

$$\begin{pmatrix} |\kappa_j|e^{-|\kappa_j|} & -|\kappa_j|e^{|\kappa_j|} \\ -(c\cdot\kappa_j)^2 + |\kappa_j|(g+\sigma|\kappa_j|^2) & -(c\cdot\kappa_j)^2 - |\kappa_j|(g+\sigma|\kappa_j|^2) \end{pmatrix}\begin{pmatrix} A_n(\kappa_j) \\ B_n(\kappa_j)\end{pmatrix} = \begin{pmatrix}\frac{1}{2}T_2(-1) \\ \Phi(\kappa_j)\end{pmatrix} \qquad (A\,4)$$

where

$$\begin{aligned}
\Phi(\kappa_j) &= 2i(c\cdot\kappa_j)(\mu_{n-e_j}\cdot\kappa_j)(g+\sigma|\kappa_j|^2)\alpha_j + (ic\cdot\kappa_j)\hat{q}_j(\kappa_j) + (g+\sigma|\kappa_j|^2)\hat{r}_j(\kappa_j)\\
&\quad - \frac{(c\cdot\kappa_j)^2}{2|\kappa_j|}T_1(0) - \frac{(g+\sigma|\kappa_j|^2)}{2}T_1(0).
\end{aligned}$$

Since $\Lambda_\sigma(c,\kappa_j)=0$ the matrix on the left hand side of (A 4) is singular and thus there is a compatibility condition required for solvability. This condition reads

$$\Phi(\kappa_j) + \frac{(c\cdot\kappa_j)^2 - |\kappa_j|(g+\sigma|\kappa_j|^2)}{2|\kappa_j|e^{-|\kappa_j|}}T_2(-1) = 0,$$

and can be written generically as $\kappa_j \cdot \mu_{n-e_j} = R_{n,j}$. From Lemma 4.8 we conclude that there exist unique μ_{n-e_j} to satisfy all of these compatibility conditions at order $|n| = \bar{n}$. We note that while w_n and v_n can be completely determined from Equation (4.9) at multi–index n, the μ_{n-e_j} require Equation (4.9) at all multi–indices $|n| = \bar{n}$ for their unique resolution. Once we have solvability we need to specify an orthogonality condition to ensure uniqueness. We choose (4.11) which requires that $\hat{v}_n(\pm\kappa_j) = 0$. From (4.9c)

$$\hat{v}_n(\kappa_j) = \frac{1}{g + \sigma |\kappa_j|^2} \left[\hat{q}_n(\kappa_j) - (ic \cdot \kappa_j)\hat{w}_n(\kappa_j, 0) - (\mu_n \cdot \kappa_j)(g + \sigma |\kappa_j|^2)\alpha_j \right]$$

and from (A 2), $\hat{w}_n(\kappa_j, 0) = A_n(\kappa_j) + B_n(\kappa_j) - \frac{1}{2|\kappa_j|} T_1(0)$, so we now have a second equation to specify $A_n(\kappa_j)$ and $B_n(\kappa_j)$ (in addition to the one from (A 4)) uniquely; the result is

$$\begin{aligned} A_n(\kappa_j) &= \frac{1}{2|\kappa_j| \cosh(|\kappa_j|)} \left[\frac{|\kappa_j| e^{|\kappa_j|}}{ic \cdot \kappa_j} \left(\hat{q}_n(\kappa_j) - (\mu \cdot \kappa_j)(g + \sigma |\kappa_j|^2) \right) \right. \\ &\qquad \left. + \frac{e^{|\kappa_j|}}{2} T_1(0) + \frac{1}{2} T_2(-1) \right] \\ B_n(\kappa_j) &= \frac{1}{2|\kappa_j| \cosh(|\kappa_j|)} \left[\frac{|\kappa_j| e^{-|\kappa_j|}}{ic \cdot \kappa_j} \left(\hat{q}_n(\kappa_j) - (\mu_n \cdot \kappa_j)(g + \sigma |\kappa_j|^2) \right) \right. \\ &\qquad \left. + \frac{e^{-|\kappa_j|}}{2} T_1(0) - \frac{1}{2} T_2(-1) \right]. \end{aligned}$$

Now, using the estimates given in Lemma 4.9 it can be shown that for all $k \in \Gamma'$

$$\begin{aligned} \|\hat{w}_n(k,y)\|^2_{L^2(dy)} &\le C \left[\langle k \rangle^{-5} |\hat{q}_n(k)|^2 + \langle k \rangle^{-3} |\hat{r}_n(k)|^2 + \langle k \rangle^{-4} \|\hat{p}_n(k,y)\|^2_{L^2(dy)} \right] \\ \|\partial_y \hat{w}_n(k,y)\|^2_{L^2(dy)} &\le C \left[\langle k \rangle^{-3} |\hat{q}_n(k)|^2 + \langle k \rangle^{-1} |\hat{r}_n(k)|^2 + \langle k \rangle^{-2} \|\hat{p}_n(k,y)\|^2_{L^2(dy)} \right] \\ |\hat{v}_n(k)|^2 &\le C \left[\langle k \rangle^{-4} |\hat{q}_n(k)|^2 + \langle k \rangle^{-4} |\hat{r}_n(k)|^2 + \langle k \rangle^{-5} \|\hat{p}_n(k,y)\|^2_{L^2(dy)} \right] \end{aligned}$$

which give the desired estimates on w_n and v_n. □

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