The Water-Wave Problem and its Long-Wave and Modulational Limits*

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Abstract

We review some of the basic mathematical results on solutions of the water-wave problem, such as existence of traveling waves and the well-posedness of the initial value problem. We also discuss rigorous results which concern the various asymptotic scalings that lead to the principal canonical equations of mathematical physics.

1 Equations of Motion

The water-wave problem refers to the motion of a three-dimensional, incompressible, inviscid fluid with a free surface $\Sigma(t) = \{z = \eta(x,t), x = (x_1, x_2) \in \mathbf{R}^2\}$ satisfying the Euler equations

$$\partial_t u + (u \cdot \nabla) u = -\nabla p - g\ e_3 \tag{1.1}$$

$$\nabla \cdot u = 0 \tag{1.2}$$

where e_3 is the unit vertical vector $(0,0,1)$. The fluid is under the influence of gravity g and surface tension whose strength (normalized by the density) is β. The domain of the fluid $\Omega(t) = \{(x,z),\ x \in \mathbf{R}^2,\ -h < z < \eta(x,t)\}$ is infinite in the horizontal directions and has a fixed bottom at $z = -h$ (which can be infinite).

The fluid is assumed to be irrotational, $\nabla \wedge u = 0$, so that the motion is described by a velocity potential φ that satisfies

$$\Delta \varphi = 0 \quad \text{in} \quad \Omega(t), \tag{1.3}$$

with the boundary conditions

$$\partial_z \varphi = 0 \quad \text{on} \quad z = -h, \tag{1.4}$$

and

$$\partial_t \varphi = -\frac{1}{2}(\nabla\varphi)^2 - g\eta + \beta\kappa(\eta), \tag{1.5}$$

$$\partial_t \eta = -\partial_x \varphi \cdot \partial_x \eta + \partial_z \varphi, \tag{1.6}$$

on the free surface $z = \eta(x,t)$ over the fluid domain. Here $\partial_x = (\partial_{x_1}, \partial_{x_2})$ is the horizontal gradient, and

$$\kappa(\eta) = \frac{\eta_{x_1x_1}(1+\eta_{x_2}^2) + \eta_{x_2x_2}(1+\eta_{x_1}^2) - 2\eta_{x_1}\eta_{x_2}\eta_{x_1x_2}}{(1+|\partial_x\eta|^2)^{3/2}} \tag{1.7}$$

is the mean curvature of the free surface.

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1.1 Equations Governing the Interface Motion

Equations (1.3)–(1.6) can be reduced to a system of integro-differential equations for the values of the above functions evaluated at the free surface only. Introducing the trace $\xi(x,t) = \varphi(x,\eta(x,t),t)$ of the velocity potential φ at the surface, and we define the Dirichlet–Neumann operator G that relates ξ to the normal derivative $\partial_n\varphi$ of the potential by

$$G(\eta)\xi = \sqrt{1+|\partial_x\eta|^2}\,\partial_n\varphi\Big|_{z=\eta} = (-\partial_x\varphi\cdot\partial_x\eta + \partial_z\varphi)\Big|_{z=\eta}. \tag{1.8}$$

On the free surface $z=\eta(x,t)$, we additionally have

$$\partial_t\xi = \partial_t\varphi - \partial_t\eta\partial_z\varphi, \qquad \partial_{x_i}\xi = \partial_{x_i}\varphi - \partial_{x_i}\eta\partial_z\varphi. \tag{1.9}$$

PROPOSITION 1.1. [11] *The free surface elevation $\eta(x,t)$ and the trace $\xi(x,t)$ of the velocity potential on this surface satisfy the system*

$$\partial_t\eta - G(\eta)\xi = 0, \tag{1.10}$$

$$\partial_t\xi + g\eta + \frac{1}{2(1+|\partial_x\eta|^2)}\Big(|\partial_x\xi|^2 - (G(\eta)\xi)^2 - 2(\partial_x\eta\cdot\partial_x)G(\eta)\xi + |\partial_x\eta|^2|\partial_x\xi|^2 - (\partial_x\eta\cdot\partial_x\xi)^2\Big) - \beta\kappa(\eta) = 0. \tag{1.11}$$

In fact this system is in Hamiltonian form, as we will see in the following paragraph.

1.2 The Water-Wave as a Hamiltonian System

The water-wave problem is a Hamiltonian system, where the Hamiltonian is given by the total energy of the fluid (Zakharov [34]) and the canonical variables are given by the pair of functions (η,ξ). Let n be the dimension of the space ($n=2$ or 3) and $x=(x_1,...,x_{n-1})$. The kinetic energy is the Dirichlet integral for the fluid domain

$$K = \frac{1}{2}\int_{\mathbf{R}^{n-1}}\int_{-h}^{\eta(x,t)} |\nabla\varphi|^2 dz dx \tag{1.12}$$

and the potential energy is

$$V = \frac{1}{2}\int_{\mathbf{R}^{n-1}} g\eta^2 dx + \beta\int_{\mathbf{R}^{n-1}}(\sqrt{1+(\partial_x\eta)^2}-1)dx. \tag{1.13}$$

The total energy is $H = K+V$. Expressing the Hamiltonian in terms of the canonical variables $(\eta, \xi = \varphi(x,\eta(x,t))$,

$$H(\eta,\xi) = -\frac{1}{2}\int_{\mathbf{R}^{n-1}}\int_{-h}^{\eta(x,t)}\varphi\Delta\varphi dz dx + \frac{1}{2}\int_{y=\eta(x,t)}\varphi\nabla\varphi\cdot n dS \tag{1.14}$$

$$+\frac{1}{2}\int_{\mathbf{R}^{n-1}} g\eta^2 dx + \beta\int_{\mathbf{R}^{n-1}}(\sqrt{1+(\partial_x\eta)^2}-1)dx. \tag{1.15}$$

Since φ is harmonic, it follows that [11]

$$H(\eta,\xi) = \frac{1}{2}\int_{\mathbf{R}^{n-1}}(\xi G(\eta)\xi + g\eta^2)dx + \beta\mathrm{Area}(\eta), \tag{1.16}$$

where the operator $G(\eta)$ is the Dirichlet-Neumann operator for the fluid domain, and $\mathrm{Area}(\eta)$ is the added surface area of $\Sigma(t)$ due to the perturbation $\eta(x,t)$.
The water-wave system is rewritten

$$\partial_t\eta = \delta_\varphi H = G(\eta)\xi \tag{1.17}$$

$$\partial_t\xi = \delta_\eta H = -g\eta - \delta_\eta K + \beta\kappa(\eta)\ . \tag{1.18}$$

which expresses the Hamiltonian form of the system (1.10)-(1.11).

1.3 The Linear Problem

When linearizing the water-wave equations about the static solution $\eta = 0$, $\xi = 0$, solutions decompose into components which are harmonic waves given by

$$\eta(x,t) = e^{i(k\cdot x-\omega t)} + \mathrm{c}\cdot\mathrm{c}, \qquad \xi(x,t) = \frac{g+\beta|k|^2}{i\omega} e^{i(k\cdot x-\omega t)} + \mathrm{c}\cdot\mathrm{c} \tag{1.19}$$

where c · c stands for complex conjugate, and $\omega = \omega(|k|)$ obeys the dispersion relation

$$\omega^2 = |k|(g+\beta|k|^2)\tanh(h|k|). \tag{1.20}$$

The graph of $\omega(|k|)$ has an inflection point, which leads to the definition of a critical wavelength $\lambda_m = 2\pi(\beta/g)^{1/2}$ for the relative importance of gravity and surface tension (see, e.g., Whitham [32], Chap. 12). For perturbations whose wavelength is large compared to λ_m, surface tension is negligible, and such perturbations are referred to as *gravity waves.* In the opposite regime where the wavelength of the perturbation is small compared to λ_m, gravity is negligible and the waves are *capillary waves.* The regime where both effects are comparable corresponds to *gravity–capillary waves.* The depth h also plays a role in the distinction between gravity and capillary waves especially with regard to traveling waves. For *Bond number* $0 < \beta/gh^2 < 1/3$, both gravity and capillarity play important role, while for $1/3 < \beta/gh^2$, the capillarity has prime importance.

2 Traveling Waves

Traveling waves refer to periodic solutions which propagate uniformly without change of form. They satisfy the relation

$$\eta(x+\gamma,t) = \eta(x,t) \quad \forall\gamma\in\Gamma, \quad \partial_t\eta(x,t) = -c\partial_x\eta(x,t) \tag{2.1}$$

where $\Gamma \subset \mathbf{R}^{n-1}$ is a lattice defining the spatial period. The simplest solutions of this form are in two dimensions, bifurcating from uniform flow, and are called *Stokes waves* [32].

THEOREM 2.1. *For any spatial period, there exist non-trivial periodic traveling wave solutions of the two-dimensional water-wave problem, with or without surface tension.*

Existence of such solutions in two dimensions, without surface tension, was first proved by Levi-Civita [23] in infinitely deep water, and by Struik [28] in water of finite depth. Their method uses strongly the close connection between the water wave in two dimensions and complex analysis. With the presence of surface tension, existence theorems for traveling waves are given in Zeidler [35], Beale [4], and Jones and Toland [19].

A new proof of this result, including the case where surface tension is present, was recently obtained by Craig and Nicholls [9]. Partially for this reason, they also constructed traveling wave solutions for the three-dimensional problem. Their method is based on the formulation (1.10)-(1.11) and in particular, is independent of complex variables techniques. Using the Hamiltonian structure, traveling solutions can be characterized as stationary points of the Hamiltonian H under the constraint of fixed momentum $I_1(\eta,\xi) = \int \eta\partial_{x_1}\xi dx = a_1$, $I_2(\eta,\xi) = \int \eta\partial_{x_2}\xi dx = a_2$. The existence theorem is analogous to Weinstein's resonant Lyapunov center theorem [31]. Both H and I are invariant under the torus action by spatial translations $\alpha \in \mathbf{R}^2\backslash\Gamma = \mathbf{T}^2 \to (\eta(x+\alpha), \xi(x+\alpha))$, and a topological argument based on a $\mathbf{T}^2$ equivariant cohomological index provides a lower bound on the number of distinct critical orbits.

THEOREM 2.2. [9] *For any periodic fundamental domain $\mathbf{T}^2 \subseteq \mathbf{R}^2$, there exist non-trivial periodic traveling wave solutions of the three-dimensional water-wave problem with surface tension $\beta > 0$.*

Existence of periodic traveling wave solutions of the three-dimensional water-wave problem, without surface tension is still an open problem, due to the presence of small divisors.

3 The Initial-Value Problem

We consider the initial value problem without surface tension ($\beta = 0$). It is convenient to study the motion of the surface elevation $\Sigma(t)$ in the form

$$X(\alpha, \beta, t) = (x_1(\alpha, \beta, t), x_2(\alpha, \beta, t), z(\alpha, \beta, t)) \tag{3.1}$$

where (α, β) are the Lagrangian coordinates, i.e

$$\frac{dX}{dt} = u(X(\alpha, \beta, t), t). \tag{3.2}$$

Let $\mathbf{n}$ be the unit normal vector pointing out of the fluid domain. Taking the normal component of (1.3), one gets

$$(X_{tt} + ge_3) \cdot \mathbf{n} = -\frac{\partial p}{\partial n}. \tag{3.3}$$

In the above equation, the first term in the l.h.s is the normal component of the acceleration and the second term ins the normal component of the gravity.

PROPOSITION 3.1. *Consider solutions to the water-wave problem in a fluid domain $\Omega(t)$ which are asymptotic at $|x| \to \infty$ to an equilibrium solution. One has*

$$a = (X_{tt} + ge_3) \cdot \mathbf{n} \geq c > 0. \tag{3.4}$$

This inequality is often referred to as the Taylor sign condition. The proposition states that it is always satisfied for smooth solutions in two and three dimensions.

A simple proof of this fact was given by Caflisch and Hou. Applying the divergence operator to the Euler equation (1.1), one obtains

$$-\Delta p = |\nabla u|^2 \geq 0 \text{ in } \Omega(t) \tag{3.5}$$

Thus the pressure p is super-harmonic in $\Omega(t)$. On the other hand, p is constant at the interface (without loss of generality p =0). The strong minimum principle implies that $p(x) > 0$ inside $\Omega(t)$. As a consequence,

$$-\frac{\partial p}{\partial n} > 0 \quad \text{on} \quad \Sigma(t). \tag{3.6}$$

At infinity, where the fluid is asymptotically at rest, $-\frac{\partial p}{\partial n} \approx g$. Therefore, there is a constant c such that (3.4) holds. This proposition is a key point in the proof of the local existence theorem (Wu [33]).

Suppose that initially, the interface $\Sigma_0 = \{X = X_0(\alpha, \beta)\}$ divides $\mathbf{R}^3$ into two simply connected regions. Suppose that it is smooth and non-self-intersecting, and that the fluid is asymptotically at rest. The hypotheses are expressed in the following conditions:

(i) $|X_0(\alpha, \beta) - X_0(\alpha', \beta')| \geq C(|\alpha - \alpha'| + |\beta - \beta'|)$, (multi-dimensional cord-arc condition)

(ii) $\partial_\alpha X_0, \partial_\beta X_0 \in H^{s-1/2}$, $\partial_t X|_{t=0} \in H^{s+1/2}$, for some $s > 5/2$, (asymptotic to equilibrium surface)

(iii) $|\partial_\alpha X_0 \wedge \partial_\beta X_0| \geq \mu$. (non-degeneracy)

THEOREM 3.1. *Then there exists a finite time T depending on the initial conditions such that the interface exists for $t \in (-T, +T)$, and it satisfies $X \in C((-T,T), H^{s+1/2}(\mathbf{R}^2))$.*
Earlier works on the two-dimensional water-wave problem in Sobolev spaces are due to Yosihara [30] and Craig [8] who proved the wellposedness of the problem locally in time when the interface is a small perturbation of the water at rest. In two and three dimensions, Kano and Nishida [20] proved local existence of solutions in spaces of analytic functions, without smallness conditions on the initial interface, using the Cauchy-Kowalevska theorem [24] [25].

Using that the Taylor sign condition is true for any local solution of the water-wave problem, the result of Beale et al.[5] in two dimensions and of Hou et al. [18] in three dimensions can be restated in the following way:
THEOREM 3.2. *The linearization of the water-wave problem near an arbitrary solution is well-posed in Sobolev spaces.*

4 Asymptotic Scalings

For the purpose of asymptotic expansions, it is useful to introduce the non-dimensional coefficients $\alpha = \frac{a}{h}$ and $\delta = \frac{h}{l}$, where h is the depth of the channel, a is a typical amplitude and l a typical wavelength.

The question that we address in this section is in which precise sense solutions obtained through formal asymptotic expansions approximate the full Euler equations (1.1)-(1.2). There are several formulations of convergence for asymptotic scaling regimes. We illustrate them on the model abstract system given by a vector field which depends upon a small parameter ϵ.

$$(4.1)_\epsilon \qquad \partial_\tau z^\epsilon = V(z^\epsilon, \epsilon), \qquad z^\epsilon(x,0) = z_0(x).$$

Let us denote the flow $z^\epsilon(\tau) = \Phi^\epsilon(\tau, z_0)$. The strongest statement of approximation of solutions of $(4.1)_0$ by solutions of $(4.1)_\epsilon$ is

$$(A) \qquad \text{For } \tau \in [0, O(1)), \quad z^\epsilon(\tau) \text{ exists and } \sup_\tau \|z^\epsilon(\tau) - z(\tau)\|_X \to 0, \text{ as } \epsilon \to 0.$$

Weaker statements either exhibit a time of existence dependent upon ϵ,

$$(A') \qquad \text{For } \tau \in [0, O(\epsilon)), \quad z^\epsilon(\tau) \text{ exists and } \sup_\tau \|z^\epsilon(\tau) - z(\tau)\|_X \to 0, \text{ as } \epsilon \to 0,$$

or express the degree to which the solutions z^ϵ is an approximate of $(4.1)_0$

$$(B) \quad \text{For } \tau \in [0, O(1)), \ z^\epsilon(\tau) \in X \text{ exists and } \sup_\tau \|\partial_\tau z^\epsilon(\tau) - V(z^\epsilon, 0)\|_{X_1} \leq O(\epsilon^p)\|z_0\|_X,$$

or express the degree to which the solution z^0 of $(4.1)_0$ is an approximate solution of $(4.1)_\epsilon$,

$$(C) \qquad \text{For } \tau \in [0, O(1)), \quad z(t) \text{ exists and } \sup_\tau \|\partial_\tau z - V(z,\epsilon)\|_{X_1} \leq O(\epsilon^p)\|z_0\|_X,$$

with $X \subset X_1$.

4.1 The shallow water limit

The shallow-water limit corresponds to the case where δ is small, while α remains of order one. Friedrichs [16] proposed a Taylor expansion of the solution in powers of δ in the form

$$(4.2) \qquad \xi^\delta = \sum_0^\infty \delta^{2n} \xi^n, \qquad \eta^\delta = \sum_0^\infty \delta^{2n} \eta^n,$$

where the coefficients $(\partial_x\xi^n, \eta^n)$ satisfy hyperbolic systems. In particular, $(\partial_x\xi^0, \eta^0)$ is solution of

$$\partial_t\xi^0 + \frac{1}{2}(\partial_x\xi^0)^2 + \eta^0 = 0, \quad \partial_t\eta^0 + \mathrm{div}(\eta^0\partial_x\xi^0) = 0. \tag{4.3}$$

A rigorous justification of Friedrichs expansion is given in [20] [22]. It is a convergence theorem in the sense (A), with X being the space of analytic functions B_ρ equipped with the norm $\|u\|_\rho = \|(1+|k|^2)^{1/2}e^{2\pi\rho|k|}\hat{u}(k)\|_{L^2}$, where $\hat{u}$ is the Fourier transform of u.

THEOREM 4.1. *Let the initial conditions $(\xi_0^\delta, \eta_0^\delta)$ be in B_{ρ_0}. For any $\rho < \rho_0$, the solution $(\xi^\delta, \eta^\delta)$ of the water-wave problem is approximated by (ξ^0, η^0), solution of (4.3), in the sense that for all $t < a(\rho_0 - \rho)$,*

$$\|\partial_x\xi^\delta - \partial_x\xi^0\|_\rho = O(\delta^2), \qquad \|\partial_x\eta^\delta - \partial_x\eta^0\|_\rho = O(\delta^2). \tag{4.4}$$

In addition, $(\xi^\delta, \eta^\delta)$ is infinitely differentiable with respect to δ, with $0 \le \delta \le 1$.

4.2 The weakly nonlinear shallow water limit

We discuss here to the two-dimensional water-wave problem in the long-wave, small amplitude limit, in the case where the nonlinearity and the dispersion exactly balance: $\alpha = \delta^2 = \epsilon$. Under the scaling

$$x' = \epsilon^{1/2}x, \quad t' = \epsilon^{1/2}t, \quad X_1' = \epsilon^{-1/2}X_1; \quad X_2' = \epsilon^{-1}X_2, \tag{4.5}$$

where $(x + X_1(x,t), X_2(x,t))$ is the Lagragian coordinate of the free surface, the quantity $q(x',t') = -\partial_{x'}X_1'(x',t')$, is governed by the Boussinesq equation $q_{t't'} - q_{x'x'} - \frac{\epsilon}{3}q_{x'x'x'x'} - \frac{3}{2}\epsilon(q^2)_{x'x'} = 0$. An additional translation $y = x' - t'$ and slowing of the time variable $t'' = \epsilon t'$, formally lead in the new variables to the Korteweg-de Vries equation $q_{t''} + 6qq_y + q_{yyy} = 0$.

The article [8] gives a rigorous justification of the derivation of the Boussinesq and KdV equations. Because both of these equations are posed in a regime describing slow time variations, comparisons of the evolution of water waves and solutions of the asymptotic limiting equations require an existence result for (1.3)-(1.6) valid over long time intervals. Introducing the space and time scaling of (4.5), there is a long time existence theorem for solutions with Sobolev-class initial data.

THEOREM 4.2. *There exists a constant Λ_0 such that for initial data $(X_1^0, X_{1t'}^0, X_2^0)(x')$ satisfying*

$$|X^0|_{C^1} + \|(X_1^0, X_{1t'}^0, X_2^0)(x')\|_{H^4} < \Lambda_0, \qquad \|(X_1^0, X_{1t'}^0, X_2^0)(x')\|_{H^{r+1}} < +\infty \tag{4.6}$$

then there is a time interval $[-T', T']$, $T' > C/\epsilon$, in which there exists a unique solution

$$(X_1, X_2)(x', t') \in C([-T', T']; H^{r+1/2} \times H^r) \cap C^2([-T', T']; H^{r-1/2} \times H^{r-1}) \ . \tag{4.7}$$

and the solution satisfies $\|(X_1, X_2)\|_{H^r} < K$, a bound which is independent of ϵ.

Because of the arbitrary freedom in Lagrangian variables to parametrize the initial fluid domain, there is an additional compatibility condition that X must satisfy. Define the Boussinesq operator to be

$$B(Y) = \partial_{t'}^2 Y - \partial_{x'}^2 Y - \frac{\epsilon}{3}\partial_{x'}^4 Y + \frac{3}{2}\epsilon\partial_{x'}^2(Y^2) \ . \tag{4.8}$$

THEOREM 4.3. *Assume that the initial data satisfies (4.6) for $r > 7$, and that the following compatibility condition holds for the initial parametrization*

$$\|(X_2^0 + X_{1x'}^0 + \frac{\epsilon}{3}X_{1x'x'x'}^0 - \epsilon(X_{1x'}^0)^2)\|_{H^{r+3}} < \epsilon^2 C_r \ . \tag{4.9}$$

Then the solution (X_1, X_2) *satisfies*

$$\|B(-\partial_{x'}X_1)\|_{H^{r-7}} \le \epsilon^2 C_r \|(X_1^0, X_{1t'}^0, X_2^0)(x')\|_{H^{r+1/2}}. \tag{4.10}$$

To adapt the evolution to the KdV regime, we set $t'' = \epsilon t'$, and we anticipate a condition to be imposed upon the initial data. Consider a one parameter family of solutions $(X_1, X_2)(x', t'; \epsilon)$ which satisfy $\|(X_1^0, X_{1t'}^0, X_2^0)(x'; \epsilon))\|_{H^{r+6}} < C_r$, and which converge in H^r as ϵ tends to zero.

THEOREM 4.4. *Suppose that (4.6) holds for some* $r > 10$, *and that the higher order analogue of the compatibility condition (4.9) holds (see eq.(4) page 914 of [8]). Set* $T'' = \lim_{\epsilon\to 0} \epsilon T' > 0$. *If the initial data* $(X_1^0, X_{1t'}^0, X_2^0)(x'; \epsilon)$ *satisfies the KdV equation approximately to second order,*

$$\|(X_{1t'}^0 + X_{1x'}^0 + \frac{\epsilon}{3} X_{1x'x'x'}^0 + \frac{3}{2}\epsilon (X_{1x'}^0)^2\|_{H^{r+1}} < \epsilon^2 C_r \tag{4.11}$$

then the function $q^\epsilon(x', t'') = -\partial_{x'} X_1(x' + t'/\epsilon', t''; \epsilon) \in C^1([0, T'']; H^{r-1})$ *has a limit* $q(x', t'')$ *as* ϵ *tends to zero which is a solution of the KdV equation.*

This is an approximation result of type (A). Theorem 4.3 is not as strong; it is of type (B), due to the fact that the Boussinesq equation is quite ill-posed. Kano and Nishida [21] proves a similar result as Theorem 4.4 for the KdV equation in spaces of analytic functions, however it is of type (A') as they do not provide a long time existence theory for solutions. Recently, improvements to Theorem 4.4 were achieved by Schneider and Wayne [27]. In particular, they showed that for arbitrary initial data (in a particular weighted Sobolev space) the solution asymptotically decouples into two components, one satisfying a left-moving KdV equation and the other a right-moving KdV equation.

4.3 The modulational limit

The purpose here is to describe the weakly nonlinear dynamics of a wave train propagating at the surface of the fluid. Over short time intervals and small propagation distances, the dynamics are linear, but cumulative nonlinear interactions result in a modulation of the amplitude on large spatial and temporal scales. We thus introduce the slow variables $T = \epsilon t$ and $X = \epsilon x$, where ϵ measures the magnitude of the wave amplitude. The solution is also expanded in the form

$$\eta = \epsilon(\eta_1 + \epsilon\eta_2 + \epsilon^2\eta_3 \ldots), \quad \xi = \epsilon(\xi_1 + \epsilon\xi_2 + \epsilon^2\xi_3 \ldots), \tag{4.12}$$

(η_1, ξ_1) satisfy the linearized problem around the water at rest

$$\eta_1 = \psi(X,T)e^{i(kx_1 - \omega t)} + \text{c.c}, \quad \xi_1 = \frac{g + \beta|k|^2}{i\omega}\psi(X,T)e^{i(kx_1-\omega t)} + \text{c.c} + \phi(X,T),$$

where the amplitude now depends on the slow variables, and ϕ is a mean field that has to be retained because of the quadratic character of the water-wave equations [29]. A multiple-scale analysis performed under the above assumptions leads to the Davey-Stewartson (DS) system, which, after rescaling of the dependent and independent variables, takes the form

$$i\psi_\tau + \sigma_1\psi_{Z_1Z_1} + \psi_{X_2X_2} + (\sigma_2|\psi|^2 + \phi_{Z_1})\psi = 0 \tag{4.13}$$

$$\alpha\phi_{Z_1Z_1} + \phi_{X_2X_2} = -b(|\psi|^2)_{Z_1}. \tag{4.14}$$

In the above system, $\tau = \epsilon^2 t$, $Z_1 = \epsilon(x_1 - \omega'(k)t)$, $X_2 = \epsilon x_2$ and σ_1, σ_2, α and b are constant coefficients. The original derivations of the modulation equations for three-dimensional water-wave problem appeared in [6] and [13] in the case of pure gravity waves. The effect of surface tension was analyzed in [14] and [1]. When the wave is modulated in the direction of propagation only, the long-time large-scale dynamics of the envelope is governed by the cubic Schrödinger equation in one space dimension [34] [17].

Let us rewrite the system (1.10)-(1.11) in the form $\mathbf{W}(\eta, \xi) = 0$.

THEOREM 4.5. [12] [10] *Let (ψ, ϕ) denote a solution of the DS system. It defines through the modulation expansion an approximate solution $(\tilde{\eta}, \tilde{\xi})$ of the water-wave problem that during a time interval $I = [0, \epsilon^{-2}\tau_1]$ with τ_1 finite and satisfies the error estimate:*
(i) *For $\alpha > 0$,*

$$\sup_{t\in I} |\mathbf{W}(\tilde{\eta}, \tilde{\xi})|_{s,q} \le M\epsilon^{4-2/q}, \tag{4.15}$$

where the constant M is bounded when the supremum over $t \in I$ of $|\psi|_{C^{s+3}}$, $|\phi|_{C^{s+2}}$, $\|\psi\|_{s+6,q}$, and $\|\phi\|_{s+6,q}$ are bounded.
(ii) *When $\alpha < 0$, let $\bar{\eta} = \tilde{\eta}(\psi, \chi\phi)$ and $\bar{\xi} = \tilde{\xi}(\psi, \chi\phi)$, where the mean field has been truncated at large distance by the cut-off function χ centered at the origin. Then,*

$$\sup_{t\in I} |\chi_1 \mathbf{W}(\bar{\eta}, \bar{\xi})|_{s,q} \le M\epsilon^{4-2/q} \tag{4.16}$$

where the constant M is bounded when the supremum over $t \in I$ of $|\psi|_{C^{s+3}}$, $|\phi|_{C^{s+2}}$, $\|\psi\|_{s+6,q}$, and $\|\chi\phi\|_{s+6,q}$ are bounded, and χ_1 a $C_0^\infty(\mathbf{R}^2)$ cut-off function centered at the origin with $\mathrm{supp}(\chi_1) \subseteq \mathrm{supp}(\chi)$.

This is a convergence result of type (C). It expresses that the approximation constructed from the solution of the DS equations, defined on a time scale $O(\epsilon^{-2})$, solves the water-wave equations up to an error that for large q, scales almost like ϵ^4. This does not however demonstrate that, on a comparable time scale, it provides an accurate approximation of the solution of the water-wave problem because the latter is proved to be well-posed only on a time scale $O(\epsilon^{-1})$, a time during which the variation of the envelope is weak and only the simple translational motion of the solution at the group velocity is significant.

When rotational perturbations are allowed in the modulation analysis near water at rest, the amplitude equations have the same form as for an irrational motion [7].

4.4 The modulational limit for two-dimensional rotational flows

We consider here Euler flows with free surface which are rotational. The simplest situation is to study perturbation of a shear flow, analogous to the work of Rayleigh on the linearized stability and the subsequent papers of Fadeev [15] and Arnold [2] on the nonlinear stability of a shear flow between two fixed plates. Such a shear flow is given by $u = (U(y), 0)$ a solution of the Euler equations, where $U(y) = \partial_y \Psi_0(0, y)$. The classical result of Rayleigh is that such a flow is linearly stable only when the shear flow has no inflection points.

The Euler equation is written in terms of the stream function Ψ ($u = \mathrm{Curl}\Psi$)

$$\partial_t \Delta\Psi + \partial_y \Psi \partial_x \Delta\Psi - \partial_x \Psi \partial_y \Delta\Psi = 0, \tag{4.17}$$

with boundary condition on the bottom $\Psi(x, -h) = 0$, and on the free surface

$$\partial_{ty}\Psi - \partial_x\eta\partial_{xt}\Psi - \partial_y\Psi\partial_{xy}\Psi + \partial_x\eta\partial_{xx}\Psi\partial_y\Psi + \partial_x\eta\partial_x\Psi\partial_{xy}\Psi = -g\partial_x\eta \tag{4.18}$$

$$\partial_t\eta + \partial_x\Psi + \partial_x\eta\partial_y\Psi = 0. \tag{4.19}$$

The linearized equations around the flow $(U(y), 0)$ have solutions, after separation of variables, of the form

$$\Psi(x, y, t) = \Psi_0(y) + Y_\sigma(y)e^{i(kx-\omega t)} + \mathrm{c}\cdot\mathrm{c}, \quad \eta(x, t) = \frac{Y_\sigma(0)}{\sigma - U(0)} e^{i(kx-\omega t)} + \mathrm{c}\cdot\mathrm{c} \tag{4.20}$$

where $\Psi_0'(y) = U(y)$, $\omega^2 = gk\tanh(hk)$, $\sigma = \omega/k$ and $Y_\sigma(y)$ is solution of

$$-Y_\sigma'' + VY_\sigma = -k^2 Y_\sigma\ , \quad -h < y < 0 \tag{4.21}$$

$$Y_\sigma'(0) - \gamma Y_\sigma(0) = 0, \qquad Y_\sigma(-h) = 0. \tag{4.22}$$

In the above equations,

$$V(y,\sigma) = \frac{U''(y)}{U(y)-\sigma}, \quad \text{and} \quad \gamma(\sigma) = \frac{1}{U(0)-\sigma}\left(U'(0) + \frac{g}{U(0)-\sigma}\right). \tag{4.23}$$

Without loss of generality, one can assume that $U(0) = 0$ and $U'(0) = A > 0$. Let $\sigma_\pm = -\frac{Ah}{2} \pm \sqrt{\frac{A^2h^2}{4} + gh}$. Using Sturm-Liouville comparison theorems, one has

THEOREM 4.6. *If $U'' > 0$, and $\sigma > \max(\sup U, \sigma_+)$, or if $U'' < 0$, and $\sigma < \min(\inf U, \sigma_-)$, then there exists a negative eigenvalue $-k^2 = -k^2(\sigma)$ with eigenfunction $Y_\sigma(y)$.*

A more elaborate comparison argument exhibits a number of additional regimes in which negative eigenvalues $-k^2(\sigma)$ exist, for certain intervals of σ, both for $U'' > 0$ and $\sigma < 0$, and for $U'' < 0$ and $\sigma > 0$.

Under the assumption that the base flow $U(y)$ is stable and monotone, modulation theory based on the ansatz

$$\Psi(x,y,t) = \Psi_0(y) + \epsilon\psi(X,T)Y_\sigma(y)e^{i(kx-\omega t)} + \mathrm{c}\cdot\mathrm{c}, \tag{4.24}$$

$$\eta(x,t) = \epsilon\frac{Y_\sigma(0)}{\sigma - U(0)}\psi(X,T)e^{i(kx-\omega t)} + \mathrm{c}\cdot\mathrm{c} \tag{4.25}$$

leads to the nonlinear Schrödinger equation for the amplitude ψ of the perturbation.

The effect of the shear flow on the stability properties of three-dimensional perturbations is studied in [26]. In two dimensions, the case of a linear shear flow was studied in [3] both analytically and numerically and it is shown that depending on the direction of propagation (along or against the shear) of the finite-amplitude waves, the effect of the shear is different.

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