Nonexistence of solitary water waves in three dimensions

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This article considers constraints on the possibility of existence of solitary water waves in three or higher spatial dimensions. In the subject of free surface water waves, the solitary waves play an important rôle in the theory of two dimensional fluid motions. These are steady solutions to the Euler equations which are localized, positively elevated above the mean fluid level, and traveling at velocities with supercritical Froude number. They provide a stable mechanism in bodies of water for transport of mass, momentum and energy over long distances. In this paper we prove that in the three (or higher) dimensional problem of surface water waves, there do not exist any localized, steady positive solutions to the Euler equations.

Keywords: fluid dynamics, water waves, solitary waves

1. Introduction

Solitary water waves have a central position in our understanding of the dynamics of water waves, and have historically played a valuable rôle in the development of the subject, from Russell's famous horseback observations to the elucidation of the family of KdV solitons in the context of an infinite dimensional algebraically integrable Hamiltonian system. The rigorous mathematical theory of solitary waves for two dimensional water waves dates to the existence theory of Friedrichs & Hyers (1954), and it includes the work of Amick & Toland (1981) and Amick, Fraenkel & Toland (1982) on the Stokes conjectures for the solitary waves of extremal form, the symmetry results of Craig & Sternberg (1988), and the results of Plotnikov (1991) on their bifurcation and multiplicity at large amplitude. In this paper we prove a theorem concerning the solitary wave problem for three (and higher) dimensional free surface flows, which in effect rules out the possibility of families of three dimensional solutions playing the same rôle as in the two dimensional case. In brief, the result is that any sufficiently well localized steady solution of the water wave problem which is of positive elevation cannot satisfy the boundary condition for force balance (the Bernoulli condition) everywhere on the free surface. The argument is based on the maximum principle for elliptic partial differential equations and the Hopf boundary point lemma. In this, the methods are related to those used in Craig & Sternberg (1988). This conclusion is obvious from some points of view, as has been pointed out to the author. For one, the KP equation has no solitary wave solutions in the absence of surface tension, and this is the system which describes the long-wave asymptotic regime for water waves. It is expected to be an accurate model for free surface water waves, at least for those solutions which are small amplitude long waves with even longer variation transverse to the direction of propagation. Secondly, the linearized water wave equations about the trivial solution in three or higher dimensions possess a family of null generalized eigensolutions for arbitrary phase velocity, so solutions might not be expected to decay at infinity, but rather they should exhibit oscillatory components for large $|x|$. These arguments are however not easily developed into a rigorous proof of nonexistence.

For the two dimensional solitary water wave problem there is a well defined mathematical theory. The existence result of Friedrichs & Hyers (1954) for two dimensional solitary waves, subsequently readdressed by Beale (1977), produces a bifurcation branch of solitary waves which are well approximated at small amplitude by the family of KdV solitons. This branch is shown by Amick & Toland (1981) and Amick, Fraenkel & Toland (1982) to be unbounded in the $C^{1,\alpha}$ topology, with its limits at infinity forming Lipschitz crests of opening angle $2\pi/3$, the Stokes waves of extremal form. All two dimensional solitary water waves are *a priori* of positive elevation above their asymptotic limits, symmetric about a single crest, and monotone decreasing on either side of this crest, as shown by Craig & Sternberg (1988). All of the above results involve a two dimensional fluid region with finite depth; indeed Sun (1997) has shown that in infinitely deep water, no two dimensional solitary water waves exist.

In the present paper, the property of positivity of the free surface elevation is not a conclusion, as it is in two dimensions, but rather it is a hypothesis of the theorem. It would be nice to improve this situation, but at present the author has not succeeded in finding a proof of an *a priori* estimate that the free surface must positive. An outcome of the analysis of this paper is that a solitary wave solution cannot be everywhere negative. It leaves open the tantalizing possibility of three dimensional solitary wave solutions which change sign, which however decay at infinity.

This paper includes a section on two dimensional solitary waves, which presents two results. The first is an alternate proof of the fact that solitary water waves are *a priori* of positive elevation alone. The original proof of this fact appears in Craig & Sternberg (1988). The second result is a nonexistence theorem for solitary waves in infinitely deep water. It again is a second proof of a fact that appears in Sun (1997) in the more general context of interfacial solitary waves with surface tension. The proof in the present paper is essentially an outcome of the results in arbitrary dimensions, and as well it serves to highlight the contrast between the two and the n dimensional theory in the direction of *a priori* positivity of solutions; it is therefore natural to include it here.

2. Results

The classical equations for the motion of the free surface of a body of ideal fluid under the influence of gravity are stated in terms of the velocity potential Φ, which we will assume to be written in a reference frame in which the solution is steady. Let $S(\eta) = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : -h < y < \eta(x)\} \subseteq \mathbb{R}^n$ be the fluid domain with free surface $\Lambda(\eta) = \{(x, \eta(x) : x \in \mathbb{R}^{n-1}\}$. We use the notation that the unit exterior normal to the free surface is $N = \sqrt{1 + |\partial_x \eta(x)|^2}^{-1}(-\partial_x \eta(x), 1)$, and that normal and (non-orthonormalized) tangential derivatives of functions defined in $S(\eta)$ at the free surface are respectively

$$\partial_N \Phi = \frac{1}{\sqrt{1+|\partial_x \eta(x)|^2}}(\partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi) \tag{2.1}$$

$$\partial_{t_j} \Phi = \partial_{x_j} \Phi(x, \eta(x)) = (\Phi_{x_j} + \partial_{x_j} \eta(x) \Phi_y)(x, \eta(x)) \ . \tag{2.2}$$

Then the Euler equations are written

$$\Delta \Phi = 0 \quad \text{within} \quad S(\eta) \tag{2.3}$$

$$\partial_N \Phi = 0 \quad \text{on} \quad \partial S(\eta) \ . \tag{2.4}$$

In addition, an extra boundary condition is imposed on the free surface component of the boundary of $S(\eta)$, which is

$$\frac{1}{2}|\nabla \Phi|^2 + g\eta = \frac{|c|^2}{2} \ , \tag{2.5}$$

where $c \in \mathbb{R}^{n-1}$ is the phase velocity. This second boundary condition (2.5) is known as the Bernoulli condition. Asymptotically as $|x| \to \infty$, the velocity potential satisfies

$$\Phi(x, y) \sim -c \cdot x \ . \tag{2.6}$$

Substituting the expression for the velocity potential

$$\Phi(x, y) = \varphi(x, y) - c \cdot x \ , \tag{2.7}$$

we obtain the equations for φ which will serve as the basis for our analysis;

$$\Delta \varphi = 0 \quad \text{in} \quad S(\eta) \tag{2.8}$$

$$\partial_y \varphi = 0 \quad \text{on the bottom} \quad y = -h \tag{2.9}$$

$$\partial_N \varphi + \frac{c \cdot \partial_x \eta}{\sqrt{1+|\partial_x \eta(x)|^2}} = 0 \ , \tag{2.10}$$

$$\frac{1}{2}|\nabla \varphi|^2 - c \cdot \partial_x \varphi + g\eta = 0 \ . \quad \text{on} \quad \Lambda(\eta) \ . \tag{2.11}$$

The main theorem of this paper is the following.

Theorem 2.1. *Suppose that $\eta \in C^1(\mathbb{R}^{n-1})$ and $\varphi \in C^1(S(\eta))$ give a solution to the equations* (2.8)(2.9) (2.10)(2.11)*, where $\eta \geq 0$ and asymptotically $\varphi(x) \to 0$ as $|x| \to 0$. Then in fact $\eta = 0$ and $\varphi = 0$.*

Remarks: The hypotheses of Theorem 2.1 can be revised to be that (η, φ) is a finite energy solution of the equations (2.8)(2.9)(2.10) (2.11), with the same conclusion. The principal point is that this condition implies that the perturbation φ of the velocity potential Φ from that of uniform flow $-c \cdot x$ vanishes asymptotically as $|x| \to \infty$. From the point of view of this paper, the difference between the case $n = 2$ and the other $n \geq 3$ is in the possible behavior of the velocity potential at infinity. In two dimensions, the asymptotic behavior of the velocity potential is that $\Phi \sim -c \cdot x + d_\pm + \varphi$, where $d_\pm$ are two constants and where $\varphi(x) \to 0$ as $x \to \pm\infty$. In general the two constants $d_\pm$ of the two dimensional problem are different, as the natural compactification of neighborhoods of infinity in the domain is disconnected. In contrast, in three and higher dimensions, neighborhoods of infinity are connected, and finite energy solutions have the asymptotic behavior $\Phi \sim -c \cdot x + d + \varphi$ for one constant d. Of course the physical equations only depend upon the gradient of Φ, so in three and higher dimensions one can set $d = 0$ without loss of generality. However for the two dimensional problem one may only set to zero one of the two of $d_\pm$ by a change of constant, and finite energy solutions needn't satisfy $\varphi(x) \to 0$ for $|x| \to 0$. In fact nontrivial solutions do not.

A second remark has to do with the hypothesis that $\eta \geq 0$. The positive elevation of the free surface of nontrivial two dimensional solitary waves is the conclusion of an *a priori* estimate of Craig & Sternberg (1988), while in Theorem 2.1 it is a hypothesis. Perhaps it is natural in one sense to seek positive solutions of the equations (2.8)(2.9) (2.10)(2.11), and one often thinks of the solitary wave phenomenon as that of a class of positive localized bump-like solutions propagating without change of form. On the other hand, we have not been able to rule out the possibility of a three dimensional solitary wave profile which changes sign. The two dimensional *a priori* estimate depends on the existence of the stream function, and the analogous proof fails in three or higher dimensions for lack of a sufficiently good replacement for the harmonic conjugate of Φ.

3. Proof of Theorem 2.1

We shall start this section with a side result, which shows that any nontrivial solution of (2.8)(2.9) (2.10)(2.11) such that $\varphi \to 0$ as $|x| \to \infty$ cannot be everywhere negative. For this, let us define the subsets $\Lambda_+(\eta) = \{x \in \mathbb{R}^{n-1} : \eta(x) > 0\}$ and $\Lambda_-(\eta) = \{x \in \mathbb{R}^{n-1} : \eta(x) < 0\}$.

Lemma 3.1. *The set* $\Lambda_+(\eta) = \{x \in \mathbb{R}^{n-1} : \eta(x) > 0\} \neq \emptyset$, *unless the solution is identically zero.*

Proof. If φ is not identically zero, then it must achieve its extrema in the closure of the fluid domain $\overline{S(\eta)}$, as $\varphi \to 0$ for $|x| \to 0$. The extrema could be minima, maxima, or both could occur. Since φ is harmonic, the strong maximum principle implies that the extrema must occur on the free surface $\Lambda(\eta)$ or on the bottom $\{y = -h\}$.

In fact the extrema cannot occur on the bottom boundary. For at a purported maximum $(x_M, -h)$, the Hopf boundary point lemma implies that $\partial_y \varphi(x_M, -h) < 0$, with strict inequality (respectively $\partial_y \varphi(x_m, -h) > 0$ at a minimum $(x_m, -h)$, with strict inequality). However the boundary conditions that (2.9) imposes are that $\partial_y \varphi(x, -h) = 0$, which is incompatible with these two possibilities.

At an extremum $P = (p, \eta(p))$ on the free surface $\Lambda(\eta)$, the Hopf lemma implies that $|\partial_N \varphi(P)| > 0$. In fact given any $Q = (q, \eta(q))$ a critical point on the free surface, then

$$\partial_{t_j} \varphi(Q) = \partial_{x_j} \varphi(Q) + \partial_{x_j} \eta(q) \partial_y \varphi(Q) = 0 \ ,$$

therefore $\partial_{x_j}\varphi(Q) = -\partial_{x_j}\eta(q)\partial_y\varphi(Q)$, and we have

$$\begin{aligned} c\cdot\partial_x\varphi(Q) &= -c\cdot\partial_x\eta(q)\partial_y\varphi(Q) && (3.1)\\ \partial_N\varphi(Q) &= \frac{1}{\sqrt{1+|\partial_x\eta|^2}}(\partial_y\varphi(Q) - \partial_x\eta\cdot\partial_x\varphi(Q)) && (3.2)\\ &= \sqrt{1+|\partial_x\eta|^2}\partial_y\varphi(Q)\ . && (3.3)\end{aligned}$$

Using these two expressions, we find

$$c\cdot\partial_x\varphi(Q) = -\frac{c\cdot\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}}\partial_N\varphi(Q)\ . \tag{3.4}$$

The kinematic condition (2.10) reads that

$$\partial_N\varphi = -\frac{c\cdot\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}}\ ,$$

therefore at critical points Q of the potential φ on the free surface, and in particular at its extremal points P, expression (3.4) states that

$$c\cdot\partial_x\varphi(Q) = |\partial_N\varphi(Q)|^2\ . \tag{3.5}$$

This information is used in the Bernoulli condition (2.11), at the extremal points P. Indeed at any critical points Q of φ, the tangential partial derivatives satisfy $\partial_t\varphi(Q) = 0$ and therefore $|\nabla\varphi(Q)|^2 = |\partial_N\varphi(Q)|^2$. Using this expression in (2.11) at Q, we have

$$\tfrac{1}{2}|\partial_N\varphi(Q)|^2 = g\eta(q)\ . \tag{3.6}$$

As mentioned above, when this identity is invoked at a critical point P, the LHS is strictly positive due to the Hopf lemma. Therefore at P the free surface satisfies $\eta(p) > 0$, which is to say the points p are in $\Lambda_+(\eta)$. This is the result of the lemma. □

The following lemma, which leads to the main result of Theorem 2.1, concerns an auxiliary harmonic function to the velocity potential or its perturbation φ. We note that the hypothesis in the $n > 2$ dimensional solitary wave problem is that the potential vanishes as $|x| \to \infty$.

Lemma 3.2. *The function defined by $\psi(x,y) = c\cdot\partial_x\varphi(x,y)$ is harmonic in the fluid domain $S(\eta)$ and it satisfies Neumann boundary conditions on the bottom $\{y = -h\}$. Suppose that the potential function satisfies $\varphi \to 0$ as $|x| \to \infty$. Then unless ψ is identically zero, it takes on both positive and negative values on the free surface $\Lambda(\eta)$.*

Proof. We observe that the operators Δ and $c\cdot\partial_x$ commute, and the latter gives a vector field tangent to the bottom boundary. Since the function φ is harmonic and since it satisfies Neumann bottom boundary conditions, then clearly

$$\Delta\psi = c\cdot\partial_x(\Delta\varphi)\ , \qquad \partial_N\psi(x,-h) = c\cdot\partial_x(\partial_N\varphi(x,-h) = 0\ .$$

This proves the first two of the assertions of the lemma.

Consider the case that $\varphi(x,y)$ and $\eta(x)$ are assumed to vanish as $|x| \to \infty$. If ψ is not identically zero we will show that it must take on both positive and negative values within the fluid domain $S(\eta)$. Indeed, consider any horizontal line $\ell = \{a+sc : a \in S(\eta), c \in \mathbb{R}^{n-1}, s \in \mathbb{R}\}$ with tangent parallel to the phase velocity c, which lies entirely within the fluid domain $S(\eta)$ (such as quite close to the bottom). The integral of $\psi(x,y)$ along ℓ must vanish, since

$$\int_{-\infty}^{+\infty} \psi(a+sc)\mathrm{d}s = \int_{-\infty}^{+\infty} c \cdot \partial_x \varphi(a+sc)\mathrm{d}s = \int_{-\infty}^{+\infty} \frac{d}{ds}\varphi(a+sc)\mathrm{d}s = 0 \ .$$

Therefore along any such line the function ψ must either change sign or vanish identically. The set of such horizontal lines ℓ is open. If ψ vanished identically on every ℓ, then by unique continuation it must vanish identically on the fluid domain.

Consider the maximum and the minimum of ψ on the fluid region $S(\eta)$. Since $\varphi(x)$ and η vanish as $|x| \to \infty$, an argument using the Poisson kernel will show that the function $\psi(x,y)$ also goes to zero as $|x| \to \infty$. Because ψ is harmonic and vanishes as $|x| \to \infty$, the maximum and minimum are achieved on the boundary $\partial S(\eta)$. An extremum P of a nonzero ψ cannot lie on the bottom boundary because, as argued before, the Hopf lemma would imply that $\partial_N \psi(x_e, -h) > 0$ at a maximum point $P = (x_e, -h)$ (respectively, $\partial_N \psi(x_e, -h) < 0$ at a minimum). This would violate the Neumann bottom boundary conditions. Thus the maximum, which is positive, and the minimum, which is negative, lie on the free surface $\Lambda(\eta)$, from which we conclude that the sets $\Lambda_+(\psi) = \{p \in \Lambda(\eta) : \psi(p, \eta(p)) > 0\}$ and $\Lambda_-(\psi) = \{p \in \Lambda(\eta) : \psi(p, \eta(p)) < 0\}$ are both nonempty. This is the third assertion of the lemma. □

Proof. (of Theorem 2.1). With the information of Lemma 3.2 in hand, we examine the Bernoulli free surface condition (2.11), namely

$$\frac{1}{2}|\nabla\varphi|^2 + g\eta = c \cdot \partial_x \varphi \ . \tag{3.7}$$

Recalling the hypothesis that $\eta > 0$ on the free surface $\Lambda(\eta)$, this implies that $c \cdot \partial_x \varphi > 0$ there, and thus $\Lambda_-(\psi) = \{p \in \Lambda(\eta) : c \cdot \partial_x \varphi(p, \eta(p))) < 0\}$ is empty. However this contradicts the conclusion of Lemma 3.2. □

4. Results for two dimensional solitary water waves

A number of results for the two dimensional problem of solitary water waves can be derived with an analysis that is in the same spirit as the previous section. The first of these is an alternate derivation of the fact that solitary waves in two dimensions are *a priori* positive. The original proof of this result appears in Craig & Sternberg (1988). The second result is that in two dimensional fluid domains which are infinitely deep, $h = +\infty$, it is not possible to have solitary water waves. This result has also appeared previously, in Sun (1997), where it has quite a different proof. In fact Sun has a more general result which excludes true solitary waves in interfaces between immiscible fluids, where the interface can also exert forces due to surface tension. The present result is just for the case in which the coefficient of surface tension is zero, and the density of the upper fluid vanishes. However the proof is very straightforward, and follows directly from the lemmata of Section 3. We first present the result of *a priori* positivity of solitary waves in two dimensions.

Theorem 4.1. *Suppose that $n = 2$ and that $h < +\infty$ in the system of equations (2.8)(2.9)(2.10) (2.11). If $(\varphi(x, y), \eta(x))$ is a C^1 solution such that $c^2 > gh$ (that is the Froude number c^2/gh is supercritical), then either $\eta(x) > 0$ for all $x \in \mathbf{R}$, or else $(\varphi, \eta) = 0$.*

Proof. The argument is based on the equations (2.3) for the velocity potential Φ. Its harmonic conjugate function, or the stream function Ψ satisfies

$$|\nabla\Psi(x, y)|^2 = |\nabla\Phi(x, y)|^2 \ , \qquad \Psi \sim cy \quad \text{as} \quad x \to \pm\infty \tag{4.1}$$

$$\Psi(x, -h) = -ch \ , \qquad \Psi(x, \eta(x)) = 0 \tag{4.2}$$

and by the maximum principle, $-ch < \Psi < 0$ within the fluid domain $S(\eta)$.

We will assume that there is a global minimum point $Q = (q, \eta(q))$ of the free surface $\Lambda(\eta)$, with η taking on the value $\eta(q) = m$. We will assume that $\eta(q) \le 0$, and as well the free surface does not touch the bottom, meaning that $-h < m \le 0$. Then $\partial_x\eta(q) = 0$, $\partial_N\Phi(Q) = \partial_y\Phi(Q) = 0$ from the kinematic condition (2.4), and from the Bernoulli condition (2.5) we deduce that

$$\frac{1}{2}|\nabla\Phi|^2 = \frac{c^2}{2} - gm$$

takes on its global maximum on the free surface. Therefore the gradient of the stream function also takes on its maximum modulus at Q,

$$\frac{1}{2}|\nabla\Psi|^2 = \frac{c^2}{2} - gm \ . \tag{4.3}$$

Compare the stream function Ψ with the linear functions

$$\Psi_0(x, y) = \frac{ch(y - m)}{h + m} \ . \tag{4.4}$$

On the bottom boundary, the two functions agree, $\Psi(x, -h) = -ch = \Psi_0(x, -h)$, as do they at the minimum point Q, at which they both vanish. Furthermore, at all points of the top boundary for which $\eta(x) > m$, the stream function satisfies $\Psi(x, \eta(x)) = 0$ while

$$\Psi_0(x, \eta(x)) = \frac{ch(\eta(x) - m)}{h + m} > 0 \ . \tag{4.5}$$

Therefore $\Psi_0 \ge \Psi$ on the boundary of $S(\eta)$, and by the maximum principle $\Psi_0 > \Psi$ throughout the fluid domain, or else the two functions coincide whereupon $\eta(x)$ is a constant. Let's proceed by assuming that they differ. At the minimum points Q on the free surface we have that Ψ and Ψ_0 are equal, a situation in which the Hopf lemma states that $\partial_N\Psi(Q) > \partial_N\Psi_0(Q)$, giving the inequality

$$\partial_y\Psi(Q) > \partial_y\Psi_0(Q) = \frac{ch}{h + m} \ . \tag{4.6}$$

Use this strict inequality in the expression (4.3), and use the estimate that $ch/(h+m) > 0$, to deduce that

$$\frac{1}{2}\left(\frac{ch}{h+m}\right)^2 < \frac{c^2}{2} - gm \ . \tag{4.7}$$

Setting $s = m/h$, the requirement (4.7) is that we satisfy

$$\left(\frac{1}{1+s}\right)^2 + \frac{2gh}{c^2}s < 1 \ . \tag{4.8}$$

However when $c^2/gh > 1$ this rational inequality has no solutions for $s \in [-1, 0]$, which is just the interval in which we are asking $s = m/h$ to lie. Therefore solutions are everywhere positive, or else they vanish identically. □

The final result is that of the nonexistence of solitary wave solutions in water of infinite depth.

Theorem 4.2. *Consider the case of* $n = 2$ *and* $h = +\infty$ *in the system of equations* (2.8)(2.10)(2.11) *where* $\eta(x) \to 0$ *as* $|x| \to \infty$ *and* $\varphi(x, y) \to 0$ *as* $x^2 + y^2 \to \infty$. *The only possible solutions with* $\eta(x) \geq 0$ *(or* $\eta(x) \leq 0$*) are* $(\varphi, \eta) = 0$.

Proof. We first comment that Lemma 3.1 applies to this situation, so that it is not possible to have nontrivial solitary wave solutions for which $\eta(x) \leq 0$ for all $x \in \mathbf{R}$.

We are left to consider the possibility that $\eta(x) \geq 0$ for all $x \in \mathbf{R}$. Without the presence of a finite bottom to the fluid domain $S(\eta)$, neighborhoods of infinity of the fluid domain in two dimensions are connected sets, as we have pointed out in our discussion of the higher dimensional cases. The asymptotic behavior of the velocity potential is that $\Phi \sim -cx + d + \varphi(x, y)$ for a single constant d and for $\varphi(x, y) \to 0$ as $x^2 + y^2 \to \infty$. Therefore our hypothesis is natural, and it includes any finite energy solution to the equations (2.8)(2.10)(2.11). Form the auxiliary function $\psi(x, y) = c\partial_x \varphi(x, y)$ as in Lemma 3.2, and suppose that it does not vanish identically. It must decay to zero as $x^2 + y^2 \to \infty$ in the fluid domain, and by Lemma 3.2 it takes on both positive and negative values in the fluid domain, implying that both sets $\Lambda_+(\psi) = \{p \in \Lambda(\eta) : \psi(p, \eta(p)) > 0\}$ and $\Lambda_-(\psi) = \{p \in \Lambda(\eta) : \psi(p, \eta(p)) < 0\}$ are nonempty. However this fact is not compatible with the Bernoulli boundary condition (2.11) and the hypothesis that $\eta(x) \geq 0$, indeed

$$\frac{1}{2}|\nabla\varphi|^2 + g\eta = c\partial_x\varphi \ ,$$

which implies that $\Lambda_-(\psi) = \emptyset$. □

5. Conclusions

The principal result of this paper is that there do not exist localized solitary water waves in three or higher space dimensions, which are everywhere elevated above the asymptotic fluid level. It is generally expected that solitary waves should be positive disturbances in the free surface, but this is not a mathematically rigorous *a priori* statement, in contrast to the case of two dimensional solitary waves (Craig & Sternberg 1988). The conclusion of Theorem 2.1 leaves open the intriguing possibility of a solution to the water wave equations(2.3)(2.4)(2.5) which is localized in space, and for which the free surface $\eta(x)$ is necessarily oscillatory. One would not expect that such hypothetical solutions would have large aspect ratio in the direction transverse to its direction of motion. Such solutions should be captured by the KP asymptotic scaling regime, and it is known that the KP does not have traveling wave solutions. Rather, I would think that the only possibility left for a three dimensional traveling wave to exist is for a wedge- or arrowhead-shaped surface elevation profile, trailing an oscillatory 'wake' downstream, perhaps in two 'tails' which decays to zero for large $|x|$. This might be much like the wake of a ship, or at least the pattern caused by a localized region of pressure applied to the surface and moving with the correct phase velocity. However there would be no such applied pressure and the motion would have to be self-sustaining, and also be able to maintain the lateral coherence of the localized solution. If such a solution did exist, one would imagine that there would be a parameter family of them, governed by an amplitude parameter, and perhaps (a finite number of) other parameters. We view such a solution to be unlikely, but not completely ruled out by the mathematical results of this paper.

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