# On the Badulin, Kharif and Shrira Model of Resonant Water Waves

Walter Craig [1]

*Department of Mathematics, Brown University,*
*Providence, Rhode Island 02912*
*(craigw@math.brown.edu)*

To Vladimir Zakharov on his 60th birthday,
with best wishes for many more to follow.

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**Abstract:** This paper is a reappraisal of the Hamiltonian model derived by Shrira, Badulin and Kharif (BKS) for three dimensional nonlinear water waves. The model was introduced in [22] in an effort to describe the formation of traveling waves with crescent shaped features that arise from the instability of the Stokes wave train at moderately large steepness. There have been observations of such traveling waves in wave tank experiments by Su, Bergin, Marler and Myrick [24] and Su [25]. Some of the regimes described in these papers are of lightly breaking waves, which are asymmetric, with all crescents facing forwards. Other regimes that they observe apparently give rise to traveling waves which have asymmetric crescent shaped features facing both forwards and backwards. We show that the BKS model describes the Stokes wave train and its loss of stability at moderate amplitudes as a Hamiltonian saddle-node bifurcation, which corresponds to the formation of a stable three dimensional wave pattern which exhibits asymmetric crescent shaped elements. The model also produces a family of solutions homoclinic to the unstable Stokes wave train, which surrounds the orbit of crescent shaped wave patterns and which provides a mechanism for transition. Other traveling wave solutions of the BKS model having nonzero transverse momentum are good candidates for the skew wave patterns possessing characteristic hexagonal shaped structures separated by quiescent stripes which are produced to the sides of the experiments in wave tanks. The BKS model has solutions which satisfy two of the three characteristics specified in [22] for nonlinear crescent shaped waves, avoiding the introduction of a dissipative mechanism to describe features of these familiar wave patterns. The one weakness of the BKS model is that the crescent shaped wave patterns are transformed to themselves under time reversal composed with a phase shift. Therefore all of the wave patterns described by the BKS model possess forward and backward facing crescent shaped elements simultaneously, associated with alternating crests. These solutions reproduce the features of some but not all of the wave patterns in the observations of [24] and Su [25]. In the deep water case, we introduce and analyse a new and more realistic four degree of freedom Hamiltonian model of water waves which has two principal five wave interactions. While being more complicated and not completely integrable, nonetheless this model has traveling wave solutions with similar crescent shaped elements, and others with the hexagonal features of the BKS model.

## 1. Introduction:

Surface water waves are readily and informally observable in our common experience, and are a basic phenomenon which has been central to the study of nonlinear wave phenomenon for over a century. Two types of three dimensional wave forms which appear among water waves are apparently ubiquitous; forward facing crescent shaped steep water wave patterns, and hexagonal, often large aspect ratio, traveling wave patterns. This paper reports on a model of three dimensional water wave evolution that was introduced by V.I. Shrira, S.I. Badulin and Ch. Kharif [22] in an effort to describe several phenomena observed in wave tank experiments performed by M.-Y. Su, M. Bergin, P. Marler and R. Myrick [24] and Su [25]. One of the observations that was reported in these papers was a class of regular and doubly periodic patterns of three dimensional crescent shaped waves. The most striking of these is of lightly breaking waves, with all of the crescent shaped features facing forwards. These solutions are apparently the result of a three dimensional instability of the two dimensional Stokes wave, which occurs at moderately large steepness. Indeed, essentially two dimensional Stokes wave trains of sufficiently large wave steepness which were initiated at the paddle of the experimental wave tank facility, develop after a number of basic periods into this three dimensional pattern of crescent shaped waves. The three dimensional pattern persists for a substantial number of basic periods before radiating a skew wave pattern to the lateral sides of the wave tank, and returning to a lower amplitude (and relatively noisy) approximately two dimensional final state. Much of the literature which discusses this experiment has concentrated on the crescent shaped pattern, while we note that the second paper of Su [25] focuses on the skew wave patterns. In the present paper we present and analyse a model of nonlinear water waves which exhibits solutions with both classes of behavior.

Theoretical descriptions of the experiments of Su, Bergin, Marler and Myrick start with the numerical solutions of D. Meiron, P. Saffman and H.C. Yuen [16], who note that the experiments are close to a bifurcation phenomenon for three dimensional traveling water wave patterns. Their paper gives a number of numerical calculations of three dimensional solutions of the water wave equations, both in Eulerian coordinates and in V.E. Zakharov's Hamiltonian formulation of the problem [26]. Their results give consistent agreement between the two choices of coordinates, at approximately the observed wave steepness. However their solutions are in all cases symmetric under reflection through a plane at each wave crest orthogonal to the direction of propagation, and the crescent shaped features that they exhibit appear in both forward and backward orientation simultaneously. The subsequent paper of Saffman and Yuen [21] is cogniscent of the rôle of a basic five wave interaction in the

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formation of these wave patterns, but the solutions that they describe remain symmetric about each crest, and furthermore no stability argument is presented.

The present paper is based on a new idea that was introduced by Shrira, Badulin and Kharif in [22]. They develop the elegant point of view of Zakharov that describes water wave evolution as a Hamiltonian system with infinitely many degrees of freedom. Starting from Zakharov's Hamiltonian formulation for Euler's equations [26], and posing the basic equations in complex symplectic coordinates, Badulin, Kharif and Shrira first subject the system to several canonical transformations (renormalization transformations) to achieve a normal form. Their model describing three dimensional crescent wave formation is derived directly from the Hamiltonian in fifth order Birkhoff normal form, which is then truncated to include only the evolution of the complex amplitudes for a principal Fourier wave number $k^{(1)}$, and two side modes $k^{(2)}$ and $k^{(3)}$ in symmetric position, which are in a fifth order resonance with $k^{(1)}$. Badulin, Kharif and Shrira assume that most of the basic phenomena of nonlinear traveling water waves are described by the interactions between these three complex Fourier modes.

However, after a preliminary analysis of this model, Shrira, Badulin and Kharif concluded that crescent patterns did not occur in the Hamiltonian equations for water waves, and that dissipation at some scale was crucial for their formation. This paper [22] introduces two dissipative modifications of their original model for this purpose. It is not unreasonable to speculate that dissipation may play a rôle, as some of the observations of Su et. al. [24] and Su [25] are of crescent-shaped patterns on waves which are lightly breaking. However the thesis of the present paper is that three-dimensional crescent-shaped waves occur in smooth free surface flows described by the Hamiltonian model of Badulin, Kharif and Shrira (BKS) itself. Indeed a more in depth look at this Hamiltonian model reveals that a number of classes of traveling wave solutions exist, some of them exhibiting relatively large steepness and strong periodic crescent-shaped features that are some of the hallmarks of the observed wave patterns of [24] and [25]. These solutions are of permanent form, and in particular they exist without the presence of breaking or other mechanisms of dissipation. Additionally there are time dependent solutions of the model which are homoclinic to a two-dimensional traveling wave pattern. These solutions start very near to the two-dimensional traveling wave, and then evolve to pass nearby to a crescent-shaped wave of permanent form, remaining in a neighborhood over many basic periods before returning asymptotically to the two-dimensional traveling wave. Recent observations by Collard and Caulliez [2] of wave patterns in a wave tank at Luminy, which are formed of crescent-shaped waves which are not breaking, are consistent with solutions having these properties. Additionally we present a class of solutions with nonzero transverse momentum which are candidates for the skew wave patterns also described in the observations of [24] and [25]. These have a strong tendency toward a (non-symmetric) hexagonal form which is characteristic of wave patterns which are principally composed of two Fourier components with wavenumbers at an acute angle to each other, and they are related to the observations and KP modeling of J. Hammack, N. Scheffner and H. Segur [10] and J. Hammack, D. McCallister, N. Scheffner and H. Segur [11] in nonresonant settings.

Our principal findings for the Hamiltonian BKS model and the four mode model of resonant water waves are as follows. (i) It is a completely integrable, three degree of freedom Hamiltonian system. The integrals can be taken to be the two components of the horizontal momentum $I_1$ and $I_2$, a well as the total energy $H_{BKS}$ itself. Action-angle variables correspond to rotating coordinates, in which stationary points correspond to traveling wave solutions. (ii) For transverse momentum $I_2 = 0$ the model possesses a solution $\eta_S(x - tc)$ analogous to the Stokes wave train, a nonlinear two-dimensional traveling wave solution, and this solution is a stable elliptic orbit up to a specific value of the momentum $I_1 = b_1$. (iii) When $I_1$ exceeds the value $b_1$, the two-dimensional solution $\eta_S(x - tc)$ undergoes a Hamiltonian saddle-node bifurcation, loosing its stability and acquiring one pair of hyperbolic directions, and an orbit homoclinic to $\eta_S(x - tc)$ is formed. A new stationary point $\eta_C(x - tc)$ is created, which is elliptic and therefore stable, corresponding to a traveling wave with three-dimensional character. The wave pattern described by $\eta_C(x - tc)$ occurs with a phase difference of $\pi/3$ between the central Fourier mode and the smaller amplitude side bands, and the resulting traveling wave has the character of a crescent-shaped pattern. The homoclinic orbit surrounds this stationary point $\eta_C(x - tc)$, and for momentum $I_1$ not too much larger than $b_1$, this orbit spends a time equivalent to numerous basic periods in a neighborhood of the elliptic stationary point. It thus exhibits a robust tendency to mimic the crescent-shaped patterns of the elliptic stationary point $\eta_C(x - tc)$ for appreciable lengths of time, before returning asymptotically to the two-dimensional traveling wave. (iv) For the same values of the momentum, the model has traveling wave solutions $\eta_H(x - tc)$ which are hexagonal in structure, having the character of a nonlinear superposition of two two-dimensional wave trains of equal amplitudes intersecting at an oblique angle. Similar solutions would also exist for systems out of resonance, and are related to the work of J. Hammack, N. Scheffner and H. Segur [10] and J. Hammack, D. McCallister, N. Scheffner and H. Segur [11] in the long wave regime as described by the KP equation. (v) For transverse momentum $I_2 \neq 0$ the model exhibits traveling wave solutions which give rise to (generally non-symmetric) hexagonal wave patterns. We propose these latter solutions as candidates for the skew wave patterns observed by Su [25]. The characteristic stripes occurring in these patterns which run at an oblique angle to their phase velocity vector correspond to rays drawn through the smaller amplitude side bars of the hexagonal pattern, and these can be clearly seen in numerical solutions of the full Euler equations [19].

One question as to the accuracy of the BKS Hamiltonian model has to do with the process of approximation by mode truncation. In the case of infinitely deep water we provide a partial resolution of this by employing a new model of resonant water waves in which one additional mode is included, which is involved with the modes of the BKS model via a second fifth order resonant interaction. Although more complicated, the principal features of the traveling wave solutions of this new model are roughly the same as for the BKS model, and we obtain analogous existence and stability results.

There is one principal weakness of the solutions produced by both the BKS model and the new model of resonant water waves, which has to do with the crescent shaped elements of solutions, and the fact that in actual water waves the most striking patterns have crescents which are all facing in the direction of propagation of the waves. The water wave equations are reversible in time, so that every forward facing solution profile has to be matched by another one that is facing backwards. The one that appears in physical experiments should be determined by a criterion of stability. In the BKS model, the class of solutions which are good candidates for crescent shaped waves are stable, and do not have crests which are symmetric under reflection in a plane perpendicular to the direction of propagation. However they do exhibit a symmetry in that time reversal, composed with a spatial translation, transforms one of these solutions to itself. In particular a solution of the BKS model which possesses forward facing crescent shaped features also simultaneously possesses identical but backwards facing crescent shaped features from other crests. These patterns are similar to those seen in some of the observations reported in [24][25], but they do not reproduce the observed wave patterns in which the crescents are principally facing forwards. Nevertheless, these solutions of the two resonant five wave models of the water wave problem exhibit two of the three principal features that are specified in [22] as hallmarks of nonlinear crescent shaped waves, without resorting to a mechanism of dissipation. It remains an open question whether a Hamiltonian model can also satisfy the third condition of [22], for all crescent shaped elements to be facing the same direction.

Using the predictions of these resonant water wave models, accurate three-dimensional traveling water wave calculations by Craig and Nicholls are presently under way, using spectral codes which provide convergence criteria with spectral accuracy to traveling wave solutions of the full Euler equations. These will be described in detail in the separate publication [19].

The analysis by means of a Birkhoff normal form for a Hamiltonian system is consistent with the spirit of Zakharov's elegant program, where one understands nonlinear phenomena in partial differential equations as being manifestations of analogues of Hamiltonian mechanics in the presence of infinitely many degrees of freedom. Vladimir Zakharov's work has been enormously influential over the past number of decades, as is testified by the breadth and the variety of articles in this volume dedicated to him on his sixtieth birthday. I am personally grateful for what he has taught me.

## 2. Hamiltonian description of the water wave problem:

The Euler equations for an ideal fluid in three dimensions with a free surface are the starting point in this problem, and the BKS model equations are derived from them, mostly along principles of classical Hamiltonian mechanics. In a fluid domain $S(\eta) = \{(x_1, x_2, y) \in \mathbb{R}^3, t \in \mathbb{R} : -h < y < \eta(x,t)\}$, the velocity field is given in Eulerian coordinates by the gradient of a potential function $\varphi$ which satisfies

$$\begin{aligned} \Delta\varphi &= 0\ , && \text{for } -h < y < \eta(x,t) \\ \partial_y\varphi &= 0 && \text{at } y = -h \\ \partial_t\eta &= \partial_y\varphi - \nabla_x\varphi \cdot \nabla_x\eta && \text{and} \\ \partial_t\varphi &= -g\eta - \tfrac{1}{2}\left|\nabla\varphi\right|^2 && \text{at } y = \eta(x,t)\ , \end{aligned} \tag{1}$$

where $g$ is the acceleration of gravity and $h$ is the average depth of the fluid. It is possible to have $h = +\infty$. We will impose periodic boundary over a fundamental domain $T(\Gamma) = \mathbb{R}^2/\Gamma$, where $\Gamma \subseteq \mathbb{R}^2$ is a lattice.

This problem can be written in Hamiltonian form, using the Hamiltonian functional and choice of canonical variables that was originally proposed by Zakharov [26],

$$H_Z = \tfrac{1}{2} \int_{T(\Gamma)} \int_{-h}^{\eta(x)} (\nabla\varphi(x,y))^2\, dydx + \tfrac{g}{2} \int_{T(\Gamma)} \eta^2(x)\, dx\ . \tag{2}$$

Introducing the canonically conjugate variables $(\eta(x), \xi(x) = \varphi(x, \eta(x)))$ as in [26], the Hamiltonian (2) can be rewritten in terms of $(\eta, \xi)$ in explicit form using the Dirichlet-Neumann operator $G(\eta)$, as described by Craig and Sulem in [5],

$$H(\eta, \xi) = \int_{T(\Gamma)} \tfrac{1}{2}\xi G(\eta)\xi + \tfrac{g}{2}\eta^2\, dx\ , \tag{3}$$

and the system (1) of partial differential equations is equivalent to the Hamiltonian system

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial_\eta H \\ \partial_\xi H \end{pmatrix} . \tag{4}$$

We will denote the flow of (4) by $\Phi_t(\eta, \xi)$. Since the Hamiltonian (3) is a quadratic form in $\xi$, one vector field component is $\partial_\xi H = G(\eta)\xi$. The other component $\partial_\eta H$ is more interesting, being the variation of the Dirichlet integral with respect to the domain defined by the graph of $\eta$, and the computation is related to the Hadamard variational formula [9] for the Greens function of the fluid domain $S(\eta)$. After a calculation which can be found in [4], system (4) has the form

$$\begin{aligned} \partial_t \eta &= G(\eta)\ \xi \\ \partial_t \xi &= -g\eta - \frac{1}{2\left(1 + |\nabla_x \eta|^2\right)} \Big( |\nabla_x \xi|^2 - (G(\eta)\ \xi)^2 \\ &\quad - 2(G(\eta)\ \xi) \nabla_x \xi \cdot \nabla_x \eta + |\nabla_x \xi|^2 \, |\nabla_x \eta|^2 - (\nabla_x \xi \cdot \nabla_x \eta)^2 \Big) \ . \end{aligned} \tag{5}$$

The Dirichlet-Neuman operator $G(\eta)$ maps the Hilbert spaces $H^1(T(\Gamma))$ to $L^2(T(\Gamma))$, and is analytic in its dependence upon $\eta \in C^1(T(\Gamma))$, therefore we may write it as a convergent Taylor series expansion $G(\eta) = G_0 + \sum_{m=1}^{\infty} G_m(\eta)$, where $G_m(\eta)$ is homogeneous of degree $m$ in $\eta$. In parallel, the Hamiltonian (3) has a well defined Taylor expansion

$$\begin{aligned} H(\eta, \xi) &= H_2(\eta, \xi) + \sum_{m \geq 3} H_m(\eta, \xi) \\ &= \int_{T(\Gamma)} \tfrac{1}{2} \xi G_0 \xi + \tfrac{g}{2} \eta^2 \, dx + \sum_{m \geq 3} \int_{T(\Gamma)} \tfrac{1}{2} \xi G_{m-2}(\eta) \xi \, dx \ . \end{aligned} \tag{6}$$

The linearized equations come from the quadratic term $H_2$ of the Hamiltonian, in which $G_0 = |D| \tanh(h|D|)$, using the definition $D = -i\partial_x$. From this expression we obtain the well known linear dispersion relation

$$\omega^2(k) = g|k| \tanh(h|k|) \ . \tag{7}$$

The linear Fourier modes which satisfy the conditions of periodicity over the fundamental domain $T(\Gamma) = \mathbb{R}^2/\Gamma$ have wavenumbers $k$ in the dual lattice $\Gamma'$, and these have the temporal frequencies $\omega(k)$, $k \in \Gamma'$.

Express the canonical variables $(\eta, \xi)$ in their Fourier series expansion

$$\eta(x) = \sum_{k\in\Gamma'} \hat{\eta}(k)e^{ik\cdot x} \qquad \xi(x) = \sum_{k\in\Gamma'} \hat{\xi}(k)e^{ik\cdot x} , \tag{8}$$

with the reality condition that $\overline{\hat{\eta}}(k) = \hat{\eta}(-k)$, $\overline{\hat{\xi}}(k) = \hat{\xi}(-k)$. We introduce the complex symplectic coordinates

$$z(k) = \sqrt{\frac{g}{2\omega(k)}}\hat{\eta}(k) + i\sqrt{\frac{\omega(k)}{2g}}\hat{\xi}(k) , \quad k \in \Gamma' \tag{9}$$

in which the Hamiltonian has the form

$$H(z, \overline{z}) = \sum_{k\in\Gamma'} \omega(k)|z(k)|^2 + \sum_{m\geq 3} \left( \sum_{|P|+|Q|=m} c(P,Q)z^P\overline{z}^Q \right) , \tag{10}$$

where $P = (p(k))_{k\in\Gamma'}$, $Q = (q(k))_{k\in\Gamma'} : \Gamma' \to \mathbb{N}$ are multi-indices with $p(k), q(k) \in \mathbb{N}$ as the individual exponents of the Taylor monomials, with $z^P = \Pi_{k\in\Gamma'} z^{p(k)}(k)$ and $\overline{z}^Q = \Pi_{k\in\Gamma'} \overline{z}^{q(k)}(k)$ forming the Taylor monomials themselves, of degree $|P| = \sum_{k\in\Gamma'} p(k)$ times $|Q| = \sum_{k\in\Gamma'} q(k)$, and the coefficients $c(P,Q)$ are derived from (6)(8) and (9) in the usual way. In complex sympletic coordinates, system (4) is expressed as

$$\partial_t z(k) = i\partial_{\overline{z}(k)} H(z, \overline{z}) . \tag{11}$$

The flow $\Phi_t$ of equation (10) conserves the two components of the horizontal momentum vector $(I_1, I_2)$, which is to say that $\{H, I_j\} = 0$, for $j = 1, 2$. As a consequence we have the following easily verified result.

**Proposition 1** *The only coefficients $c(P,Q)$ which are non-zero satisfy the two conditions*

$$\langle P - Q, k\rangle = \sum_{k\in\Gamma'} (p(k) - q(k))k = 0 , \tag{12}$$

*for $k = (k_1, k_2) \in \Gamma'$.*

Another basic conserved quantity is $M = \int_{T(\Gamma)} \eta \, dx = |T(\Gamma)|^{1/2}\hat{\eta}(0)$, which is the added mass. Indeed one can check that $\{M, H\} = 0$, and therefore we may and will restrict our considerations of the evolution of (4) to the subspace of Fourier series with $\hat{\eta}(0) = 0$. In a real situation this simply corresponds to a proper choice of the parameter $h$.

A monomial term $c(P,Q)z^P\overline{z}^Q$ in the Hamiltonian H in (10) is said to be *resonant* of order $m$ if its multi-indices $(P,Q)$ satisfy

$$\langle P-Q,\omega\rangle = \sum_{k\in\Gamma'}(p(k)-q(k))\omega(k) = 0 \ , \qquad |P|+|Q| = m \ , \qquad (13)$$

with $\omega = (\omega(k))_{k\in\Gamma'}$. This is of course to be taken in conjunction with the conditions (12) in order to be relevant.

**Resonant triads:** When $|P|+|Q|=3$, there are no solutions of the three equations (12) (13), as a simple geometrical argument with the dispersion relations will show [6]. If the effects of surface tension are included, then this conclusion is no longer true, and resonances are possible.

**Resonant quartets:** When $|P|+|Q|=4$, the three conditions (12) (13) for resonance do have solutions. The generic resonances $P=Q$ are of course present, although the associated resonant monomials are quadratic in the action variables $I(k)=|z(k)|^2$, and are therefore benign. A number of other resonant terms are possible, depending in a sensitive way on the depth $h$. In case $h=+\infty$, and considering only two-dimensional fluid motions, there are Benjamin-Feir resonances, corresponding to the resonant monomials of the Hamiltonian $H$ of degree four which are described by the multi-indices

$$P-Q = \delta_{qm^2} - \delta_{-q(m+1)^2} - \delta_{-qm^2(m+1)^2} + \delta_{-q(m^2+m+1)^2} \ ,$$

where $\delta_\ell$ is the multi-index with all zero entries except for that of index $\ell$, where it is 1. For each $q\in\mathbb{N}$, $m\in\mathbb{N}$ this reflects a four wave resonant interaction between the wave numbers $k^{(1)}=qm^2$, $k^{(2)}=q(m+1)^2$, $k^{(3)}=qm^2(m+1)^2$, and $k^{(4)}=q(m^2+m+1)^2$, that is, an $m : m+1 : m(m+1) : m^2+m+1$ relationship of proportionality between the temporal frequencies. It has turned out however that these are irrelevant to any dynamical questions of deep water waves, as the associated coefficients $c(P,Q)$ vanish in a Birkhoff normal forms calculation [7][6]. In the two-dimensional problem with $h=+\infty$, the two abovementioned possibilities are the only two classes of resonant quartets. In cases where $0<h<+\infty$, or in three-dimensional cases, there are other possible resonant quartets which can occur, again depending upon $h$ and the fundamental domain $T(\Gamma)$ in a sensitive manner.

**Resonant quintets:** In the case of two-dimensional fluid motion and for $h=+\infty$, the class of five wave resonant interactions has been characterized in [6] [15], and in the latter reference the Birkhoff normal for has been expressed in full. The simplest resonant term corresponds to a fifth-order $1:2$ resonance, with $P-Q=3\delta_m-\delta_{-m}-\delta_{4m}$ for any $m\in\mathbb{N}$. In [3] it is shown that this term generally leads to an instability of any sufficiently accurate finite mode approximation to the water wave problem. When $0 < h < +\infty$ there can be many possible resonant quintets.

In the case of fully three-dimensional fluid motions, the most important resonant quintet is directly related to the problem of persistent crescent-shaped wave patterns. The proposal for this originates in the papers of Su et. al. [24] and Su [25], see also Saffman and Yuen [20] and Milewski and Keller [17]. The resonance stems from one principal wave number $k^{(1)} = (k_1^{(1)}, 0)$ and its interactions with two satellites $k^{(2)} = (k_1^{(2)}, k_2^{(2)}) = (3k_1^{(1)}/2, k_2^{(2)})$ and $k^{(3)} = (k_1^{(3)}, k_2^{(3)}) = (3k_1^{(1)}/2, -k_2^{(2)})$. By construction $3k^{(1)} - k^{(2)} - k^{(3)} = 0$, and the monomial $z^3(k^{(1)})\overline{z}(k^{(2)})\overline{z}(k^{(3)})$ will be resonant when

$$3\omega(k^{(1)}) - \omega(k^{(2)}) - \omega(k^{(3)}) = 0 \ . \tag{14}$$

Given a triple $(k^{(1)}, k^{(2)}, k^{(3)})$ of wavenumbers satisfying (12) and (14), their integer linear combinations span a *resonant lattice* $\Gamma'_r$.

**Proposition 2** *For each $0 < h \leq +\infty$ there is a unique choice of the wavenumber component $k_2^{(2)}$ for which (14) is satisfied. When $h = +\infty$ then $k_2^{(2)} = (3\sqrt{5}/4)k_1^{(1)}$, corresponding to an opening angle of $\Theta_\infty = \arctan(\sqrt{5}/2)$ between $k^{(1)}$ and $k^{(2)}$ or $k^{(3)}$. This resonant opening angle decreases monotonically to zero for decreasing depth $h$.*

**Proof** Since $|k^{(2)}| = |k^{(3)}|$, then $\omega(k^{(2)}) = \omega(k^{(3)})$. The right hand side of (14) is continuous and monotone decreasing in $k_2^{(2)}$, it is positive for $k_2^{(2)} = 0$ by the concavity of the dispersion relation (7) in $|k|$, while it is negative for large $k_2^{(2)}$, hence there is a unique root. When $h = +\infty$ then $\omega(k) = \sqrt{g|k|}$, therefore (14) reads

$$3\sqrt{k_1^{(1)}} - 2\sqrt[4]{(k_1^{(2)})^2 + (k_2^{(2)})^2} = 0 \ .$$

Substituting $k_1^{(2)} = 3k_1^{(1)}/2$ and solving the resulting quartic for $k_2^{(2)}$ in terms of $k_1^{(1)}$, one obtains the value for $\Theta_\infty$ in the proposition. In the limit of small $h$, the expression $\lim_{h\to 0} \omega(k)/h$ is linear in $|k|$ and (14) is satisfied only for $k^{(2)}$ parallel to $k^{(1)}$. Furthermore using (14), we have

$$3\partial_h\omega(k^{(1)}) - 2\partial_h\omega(k^{(2)})\frac{d}{dh}k_2^{(2)} = 0 \ , \tag{15}$$

from which we deduce that $\frac{d}{dh}k_2^{(2)} = \frac{3}{2}\partial_h\omega(k^{(1)})/\partial_h\omega(k^{(2)})$. Since $\partial_h\omega > 0$ for nonzero $|k|$, the monotonicity of $\Theta$ in $h$ follows. $\square$

The monotonicity in $h$ of the opening angle of this resonant fundamental domain gives an amusing suggestion for a nonlinear inverse problem to sample the depth under certain conditions, without knowing the wave height, steepness or phase velocity (but using information that is available from, for example, an aerial photograph). Given a wavefield which is essentially two-dimensional but undergoing a transition to a three-dimensional configuration, the two-dimensional period and the principal period exhibited in the third transverse direction define an opening angle for a fundamental domain $T(\Gamma)$. With the assumption that the most obvious instability arises from the above resonance, one then determines the (unique) average depth $h$ such that (14) holds. This is a quantitative version of the observation that three-dimensional structure of crests in open water occurs with relatively short transverse wavelength, while near the shore the three-dimensional structure of crests tends to be widely spaced.

## 3. Normal Forms Transformations:

An idea that has been often used in the subject of Hamiltonian evolution equations is that of normal forms; one performs a (finite) sequence of canonical transformations of the given system in order to reduce it to a linear problem, or at least to a problem in the simplest nonlinear form. In the water waves problem, the quiescent state $z = 0$ is an elliptic equilibrium of the system, and the relevant goal is a Birkhoff normal form. At the $m$th step of this procedure, nonresonant terms of the $m$th term of the Taylor expansion about $z = 0$ of the Hamiltonian $H$ are transformed to zero, with a canonical transformation which leaves the terms of order less than $m$ invariant, and which possibly modifies terms of order $m+1$ and higher. The formal aspects of this process lead us to expect at the $m$th step, $m = 3, 4, \dots$ a near-identity canonical transformation $f^{(m)}$ which removes the nonresonant terms of the $m$th order Taylor polynomial of $H^{(m-1)} = H \circ f^{(m-1)} \circ f^{(m-2)} \dots \circ f^{(3)}$, leaving invariant the terms of $H^{(m-1)}$ of its Taylor polynomials of orders $j = 3, \dots m-1$, and possibly modifying terms of order $m+1$ or higher. After the $m$th transformation the resulting Hamiltonian $H^{(m)}$ has the form

$$H^{(m)}(z, \overline{z}) = H_2(z, \overline{z}) + \sum_{j=3}^{m} H_j^{\text{resonant}}(z, \overline{z}) + R_{(m+1)}(z, \overline{z}) \ , \tag{16}$$

with

$$H_j^{\text{resonant}} = \sum_{\substack{P,Q : |P|+|Q|=j, \\ \langle P-Q, \omega \rangle = 0 \\ \langle P-Q, k \rangle = 0}} c^{(j)}(P, Q) z^P \overline{z}^Q \ . \tag{17}$$

Unless the system is completely integrable in the action-angle variables $(I(k),$

$\theta(k) = \arg(z(k)))$ it is unlikely that the sequence of transformations $f^{(m)} \circ f^{(m-1)} \ldots \circ f^{(3)}$ converges as $m$ goes to infinity on any neighborhood of the origin $z = 0$. Furthermore Hamiltonian systems with infinitely many degrees of freedom, such as PDE and more pertinently the water wave problem, have analytic issues associated with the individual transformations $f^{(m)}$, similar to the small divisor problem, which arise in their construction at each step of the process. The analysis of the Birkhoff normal form for the two-dimensional case of the transformation $f^{(3)}$ is described in [3], but not much more is known in general. We will only consider formal aspects of the Birkhoff normal forms transformations in this paper. Discussions of normal forms transformations on a formal level for the water waves problem appear in [12] and in [1]. Related considerations (in the continuum rather than in the periodic setting) appear in [27].

## 4. The Model of Badulin, Kharif and Shrira:

Natural model problems for Hamiltonian systems which can be written in the form of (4), with Hamiltonian in Birkoff normal form to some order, are given by simply truncating the normal forms Hamiltonian

$$H(z, \overline{z}) = H_2(z, \overline{z}) + \sum_{j=3}^{m} H_j^{\text{resonant}}(z, \overline{z}) \ .$$

In many cases of interest this however remains an infinite dimensional system in its own right, and to effectively obtain qualitative information it may be expedient to restrict the infinite-dimensional problem to a finite number of variables. That is, one fixes all but finitely many Fourier modes to be $z(k) = 0$, projecting the problem onto a finite dimensional subspace of the full phase space $E_M = \{z(k) = 0, k \neq k_\ell \ : \ \ell = 1, \ldots M\}$. This method of deriving model problems for the water waves problem by restriction to finite subspaces of modes appears in the work of Stiassnie and Shemer [23], however without the Birkhoff normal forms transformations. We emphasize the fact that the subspace $E_M$ is rarely left invariant by the flow $\Phi_t(z, \overline{z})$ of the full problem, nor does the projection onto $E_M$ usually commute with the flow $\Phi_t(z, \overline{z})$, so that this truncation is not necessarily a mathematically justifiable approximation.

The derivation of the BKS model of [22] follows the recipe above, performing a formal $m = 5$ Birkhoff normal forms transformation on the three-dimensional water waves Hamiltonian, and then restricting the resulting Hamiltonian to the subspace spanned by the Fourier modes with wave numbers $k^{(1)}, k^{(2)}$ and $k^{(3)}$ of the resonant lattice $\Gamma_r'$. In [22] the authors set $h = +\infty$ although this is not necessary as long as one can perform the sometimes tedious calculations for the normal form Hamiltonians $H^{(m)}$. The result from [22] is

$$
\begin{aligned}
H_{BKS} = & \sum_{\ell=1}^{3} \omega(k_\ell)|z(k^{(\ell)})|^2 + \tfrac{1}{2}\sum_{\ell=1}^{3}\Big( V_{\ell\bar{\ell}\ell\bar{\ell}}|z(k^{(\ell)})|^4 \\
& + \sum_{\substack{j=1 \\ j\neq\ell}}^{3} V_{\ell\bar{\ell}j\bar{j}}|z(k^{(\ell)})|^2|z(k^{(j)})|^2\Big) \\
& + 6\,\mathrm{re}\left( W_{111\bar{2}\bar{3}}\ z^3(k^{(1)})\overline{z}(k^{(2)})\overline{z}(k^{(3)})\right) ,
\end{aligned} \tag{18}
$$

where the interaction coefficients $V_{\ell\bar{\ell}j\bar{j}}$ and $W_{111\bar{2}\bar{3}}$ are constants which are derived as part of the calculation. The two horizontal momenta in the space $E_3$ of restricted modes are

$$
I_1 = |z(k^{(1)})|^2 + \tfrac{3}{2}(|z(k^{(2)})|^2 + |z(k^{(3)})|^2) , \tag{19}
$$

$$
I_2 = \tfrac{3}{2}(|z(k^{(2)})|^2 - |z(k^{(3)})|^2) , \tag{20}
$$

and it is easy to check that these quantities Poisson commute with each other and with the Hamiltonian $H_{BKS}$. We repeat the remark that the Hamiltonian $H_{BKS}$ does not include all of the terms of the $m = 5$ Birkhoff normal form, and that the restricted mode space $E_3$ is not invariant under the full Euler flow $\Phi_t$, nor necessarily under the fifth order Birkhoff normal form Hamiltonian $H^{(5)}$. Indeed in the deep water case $h = +\infty$ there is a resonant monomial associated with the quintet $P - Q = 3\delta_{k^{(\ell)}} - \delta_{-k^{(\ell)}} - \delta_{4k^{(\ell)}}$ for $\ell = 1, 2, 3$, which gives rise to a non-zero component of the Hamiltonian vector field of $H^{(5)}$ orthogonal to $E_3$ [3].

From complex symplectic coordinates of the model equations (11) one introduces initial action-angle variables

$$
z(k^{(\ell)}) = \sqrt{I(k^{(\ell)})}e^{i\theta(k^{(\ell)})} , \qquad \ell = 1, 2, 3 , \tag{21}
$$

and we use the vectorial notation $I = (I(k^{(\ell)}))^T_{\ell=1,2,3}$, $\theta = (\theta(k^{(\ell)}))^T_{\ell=1,2,3}$. It is useful to introduce rotating coordinates $M = (M_1, M_2, M_3)^T$, $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T$ in a way that respects the two horizontal momenta (19);

$$
\begin{aligned}
\sqrt{\tfrac{11}{2}}M_1 = I_1 &= I(k^{(1)}) \ + \tfrac{3}{2}(I(k^{(2)}) + I(k^{(3)})) , \\
\tfrac{3}{\sqrt{2}}M_2 = I_2 &= \tfrac{3}{2}(I(k^{(2)}) - I(k^{(3)})) .
\end{aligned}
$$

Define the third action variable to be

$$
\sqrt{11}M_3 = 3I(k^{(1)}) - I(k^{(2)}) - I(k^{(3)}) , \tag{22}
$$

in this way $M = RI$ where $R$ is a rotation. Setting $\Phi = R\theta$, the transformation from $(I(k^{(\ell)}), \theta(k^{(\ell)}))_{\ell=1}^3$ to $(M, \Phi)$ is canonical and the Hamiltonian (18) in the new variables has the form

$$H_{BKS} = \sqrt{\tfrac{11}{2}}\omega(k^{(1)})M_1 + Q(M) \\ +6W_{11\overline{1}\overline{2}\overline{3}}I^{3/2}(k^{(1)})(M)\sqrt{I(k^{(2)})(M)I(k^{(3)})(M)}\cos(\sqrt{11}\Phi_3)$$

where $Q(M)$ is a quadratic form in $M$, and the original action variables $I(k^{(\ell)})(M)$ and the momenta $I_j(M)$ are considered as functions of $M$ and are left in this form for convenience in performing calculations. The transformed Hamiltonian is independent of the two angle variables $(\Phi_1, \Phi_2)$. From [1], [22], and also for phenomenological reasons, the coefficients of (18) satisfy $3V_{1\overline{1}1\overline{1}} - V_{1\overline{1}2\overline{2}} - V_{1\overline{1}3\overline{3}} = C_{BKS}(1,3) < 0$ and $W_{111\overline{2}\overline{3}} > 0$.

**Theorem 3** *The Hamiltonian system*

$$\dot{z}(k^{(\ell)}) = i\partial_{\overline{z}(k^{(\ell)})}H_{BKS}\ , \qquad \ell = 1,2,3 \tag{23}$$

*is a completely integrable three degree of freedom system, with Poisson commuting integrals $M_1, M_2$ and $H_{BKS}$.*

Define the phase plane $S(a) = \{(M_3, \Phi_3) : I_1(M) = a_1, I_2(M) = a_2\}$, ignoring the two angles $(\Phi_1, \Phi_2)$ (the Marsden - Weinstein reduction). For given $(a_1, a_2)$ with $0 \leq |a_2| \leq a_1$, the set $S(a)$ is topologically a sphere $S^2$, parametrized by $\{(M_3, \Phi_3) : 0 \leq \Phi_3 < 2\pi/\sqrt{11}, M_3^- \leq M_3 \leq M_3^+\}$, with coordinate singularities at the two poles $P_\ell = \{I(k^{(\ell)})(M) = 0\}$, $\ell = 1, 2$. The limits $M_3^\pm$, which are affine linear in $(M_1, M_2)$, result from inspection of the image of the transformation $R$, for which the variables $I(k^{(\ell)})_{\ell=1}^3$ lie in the positive orthant.

**Traveling waves:** The classical principle is that traveling wave solutions can be characterized as critical points of the energy $H_{BKS}$ under the constraint of fixed momentum $I = (I_1(M), I_2(M))$. Restricting to the phase plane $S(a)$, critical points $T$ of $H_{BKS}$ satisfy the Lagrange multiplier rule

$$\delta H_{BKS} = c_1\delta I_1 + c_2\delta I_2\ . \tag{24}$$

In the following discussion we will take $I_2 = \frac{3}{\sqrt{2}}M_2 \geq 0$, so that $I(k^{(2)}) \geq I(k^{(3)})$; since the problem is invariant under the interchange of $k^{(2)}$ and $k^{(3)}$, the result will be general.

Away from the poles $P_1$, $P_2$, (24) is equivalent to the system of two equations

$$\partial_{M_3}H_{BKS}(M_3, \Phi_3) = 0\ , \tag{25}$$

$$\partial_{\Phi_3} H_{BKS}(M_3, \Phi_3) = 0 \ . \tag{26}$$

The two coordinate singularities $P_\ell$, $\ell = 1, 2$ of the phase plane $S(a)$ will be treated separately. At a solution $T^0 = (M_3^0, \Phi_3^0)$ of (25)(26), the two components $(c_1, c_2)$ of the phase velocity are recovered by the expressions

$$\partial_{M_1} H_{BKS}(M_3^0, \Phi_3^0) = \sqrt{\tfrac{11}{2}} c_1 \ , \quad \partial_{M_2} H_{BKS}(M_3^0, \Phi_3^0) = \tfrac{3}{\sqrt{2}} c_1 \ ,$$

and clearly $H_{BKS}$ automatically satisfies $\partial_{\Phi_1} H_{BKS} = 0 = \partial_{\Phi_2} H_{BKS}$. The component (26) is explicitly

$$\partial_{\Phi_3} H_{BKS} = -6\sqrt{11} W_{111\overline{2}\overline{3}} I^{3/2}(k^{(1)})(M) \sqrt{I(k^{(2)}) I(k^{(3)})} \sin(\sqrt{11}\Phi_3), \tag{27}$$

and this expression vanishes for $\sqrt{11}\Phi_3 = m\pi$, $m \in \mathbb{Z}$ (and at the poles $P_1, P_2$). Since $\sqrt{11}\Phi_3 = 3\theta(k^{(1)}) - \theta(k^{(2)}) - \theta(k^{(3)})$ and the angle $(\theta(k^{(2)}) + \theta(k^{(3)}))$ can be set to $2\pi j$ for any $j \in \mathbb{Z}$ by an appropriate relocation of the origin of $\mathbb{R}_x^2$, distinct solutions of (27) which are different from the poles $P_1$ and $P_2$ satisfy either $\sqrt{11}\Phi_3 = 0$ or $\pi$. This corresponds to either $\theta(k^{(1)}) = 0$ or $\theta(k^{(1)}) = \pi/3$, which are the only two cases.

**Theorem 4** *Fix the phase plane $S(a)$. If $I_2(M) > 0$ the system of equations (25)(26) has solutions $T$ on $S(a)$ which are either*

*(i) at the pole $P_1 = T_1$ at which $I(k^{(1)})(M) = 0$,*

*(ii) or for angle $\sqrt{11}\Phi_3 = 0$ or $\pi$.*

*In case (ii) there are at most two distinct solutions $T_j$ for each of the choices of angle. If $I_2(M) = 0$ there is an additional solution at the pole $T_2 = P_2$ at which $I(k^{(2)})(M) = I(k^{(3)})(M) = 0$.*

**Proof** The above discussion of the phase plane has already identified the angles that are possible for (24) to be satisfied, and it has reduced the question to whether the poles $P_1$, $P_2$ satisfy (24), and to the count of the number of solutions. First of all, $\partial_{\Phi_3} H_{BKS}(P_1) = 0$, while $\partial_{M_3} H_{BKS}(P_1)$ is independent of $\Phi_3$, and this implies that $P_1$ is an elliptic stationary point for system (23), about which the linearized solutions oscillate with angular frequency $\partial_{M_3} H_{BKS}(P_1)$. This solution is principally composed of Fourier modes for wave numbers $k^{(2)}$ and $k^{(3)}$, with none of $k^{(1)}$, and it corresponds to the traveling wave solution $\eta_H(x-tc)$ which in many cases has a characteristic hexagonal spatial structure. It is easy to check (23) in local polar coordinates that when $I_2(M) > 0$, the vector field is smooth and non-stationary in a neighborhood of $P_2$. However when $I_2(M) = 0$ then $P_2$ is also a stationary point. This corresponds to the two-dimensional solution $\eta_S(x - ct)$ analogous to the Stokes water wave train, with principal Fourier component with wavenumber $k^{(1)}$. We will discuss this solution and its stability properties in more detail below.

For stationary points $T$ of $H_{BKS}$ on $S(a)$ away from the poles, then $\sqrt{11}\Phi_3 = m\pi$ and equation (25) reads that

$$\begin{aligned}\partial_{M_3} H_{BKS} = {} & \partial_{M_3} Q(M) \\ & +(-1)^m \partial_{M_3} 6\sqrt{11} W_{111\overline{23}} I^{3/2}(k^{(1)})(M)\sqrt{I(k^{(2)})I(k^{(3)})} \\ = {} & 0\ .\end{aligned} \tag{28}$$

The function $F_1(M_3) = \partial_{M_3} Q(M)$ is an affine linear function of $M_3$, and solutions of (25) correspond to intersections of the graph of the function $(-1)^{m-1} F_1(M_3)$ with that of $F_2(M_3) = 6\sqrt{11} W_{111\overline{23}}\, \partial_{M_3}(I^{3/2}(k^{(1)})(M)\, \sqrt{I(k^{(2)})I(k^{(3)})})$.

**Proposition 5** *The function $F_2(M_3)$ is concave over the domain $\{M_3 : 0 \leq I(k^{(1)})(M),\ I(k^{(3)})(M)\}$, $F_2(M_3) > 0$ for $M_3 = M_3^+$ and $F_2(M_3) = 0$ for $M_3 = M_3^-$. The latter corresponds to the pole $P_2$ at which $I(k^{(3)})(M) = 0$.*

Since an affine linear function and a concave function can intersect in at most two points, Theorem 4 follows. □

**Proof** (of Proposition 5) On the variety $I(k^{(2)})(M) = I(k^{(3)})(M)$, the function $F_2$ simplifies to be

$$F_2 = 6\sqrt{11} W_{111\overline{23}} \partial_{M_3}(I(k^{(1)})^{3/2} I(k^{(2)})) = 6W_{111\overline{23}} \sqrt{I(k^{(1)})}(\tfrac{9}{2} I(k^{(2)}) - I(k^{(1)}))\ .$$

Using that $\partial_{M_3} I(k^{(1)}) = \frac{3}{\sqrt{11}}$ and $\partial_{M_3} I(k^{(2)}) = \frac{-1}{\sqrt{11}} = \partial_{M_3} I(k^{(3)})$, we arrive at the expression

$$\partial^2_{M_3} F_2 = -\frac{6W_{111\overline{23}}}{11} \frac{1}{\sqrt{I(k^{(1)})}^3} \frac{81}{8} \left(I(k^{(2)}) + 2I(k^{(1)})\right)$$

which is negative since $W_{111\overline{23}} > 0$. In the general case one can take logarithmic derivatives of the expression $I^{3/2}(k^{(1)})\sqrt{I(k^{(2)})I(k^{(3)})}$ to find that

$$\begin{aligned}\partial^2_{M_3} F_2 = {} & 6\sqrt{11} W_{111\overline{23}} \partial^3_{M_3} \left(I^{3/2}(k^{(1)})\sqrt{I(k^{(2)})I(k^{(3)})}\right) \\ = {} & 6\sqrt{11} W_{111\overline{23}} \Bigg(-\frac{567}{8}\frac{1}{I(k^{(1)})^3} - \frac{5}{8}\frac{1}{I(k^{(2)})^3} - \frac{5}{8}\frac{1}{I(k^{(3)})^3} \\ & -\frac{45}{8}\frac{1}{I(k^{(1)})I(k^{(2)})^2} - \frac{45}{8}\frac{1}{I(k^{(1)})I(k^{(3)})^2} + \frac{297}{8}\frac{1}{I(k^{(1)})^2 I(k^{(2)})}\end{aligned}$$

$$+\frac{297}{8}\frac{1}{I(k^{(1)})^2 I(k^{(3)})}+\frac{5}{8}\frac{1}{I(k^{(2)})I(k^{(3)})^2}+\frac{5}{8}\frac{1}{I(k^{(2)})^2 I(k^{(3)})}$$
$$-\frac{27}{4}\frac{1}{I(k^{(1)})I(k^{(2)})I(k^{(3)})}\Bigg)\ .$$

The quantity $-\frac{5}{8}I(k^{(2)})^{-3}+\frac{5}{8}I(k^{(2)})^{-2}I(k^{(3)})^{-1}+\frac{5}{8}I(k^{(2)})^{-1}I(k^{(3)})^{-2}-\frac{5}{8}I(k^{(3)})^{-3}$ within the braces of (29) is factored as $-(A-B)^2(A+B)$, which is negative for $A=I(k^{(2)})^{-1}$ and $B=I(k^{(3)})^{-1}$ both positive. Hence its contribution is negative. Inspecting the remaining terms within the braces, the quantity $297I(k^{(1)})^{-2}I(k^{(2)})^{-1}$ $-45I(k^{(1)})^{-1}I(k^{(2)})^{-2}$ $-\frac{297}{90}I(k^{(1)})^{-3}$ is $-I(k^{(1)})^{-1}$ times a square, as is the analog quantity involving $I(k^{(1)})$ and $I(k^{(3)})$, so their contributions to (29) are negative. This leaves $(-\frac{567}{8}+2\frac{297}{90})I(k^{(1)})^{-3}$ $-\frac{27}{4}I(k^{(1)})^{-1}I(k^{(2)})^{-1}I(k^{(3)})^{-1}$ within the braces, but this is clearly positive, and the claim of concavity in the proposition follows. The properties of $F_2(M_3)$ at the poles $P_1$ and $P_2$ are clear from inspection. $\square$

**Stability:** Away from the poles $P_1$, $P_2$, the vector field (23) restricted to the phase plane $S(a)$ is given in the coordinates $(\Phi_3, M_3)$ by

$$\frac{d}{dt}\begin{pmatrix}\Phi_3\\ M_3\end{pmatrix}=\begin{pmatrix}\partial_{M_3}H_{BKS}\\ -\partial_{\Phi_3}H_{BKS}\end{pmatrix}\ . \tag{29}$$

The linearized equations of (29) at a stationary point $T$ which is not one of the two coordinate singularities $T_1=P_1$ or $T_2=P_2$ are given by

$$\frac{d}{dt}\begin{pmatrix}\varphi_3\\ m_3\end{pmatrix}=\begin{pmatrix}\partial_{M_3}\partial_{\Phi_3}H_{BKS} & \partial^2_{M_3}H_{BKS}\\ -\partial^2_{\Phi_3}H_{BKS} & -\partial_{M_3}\partial_{\Phi_3}H_{BKS}\end{pmatrix}\begin{pmatrix}\varphi_3\\ m_3\end{pmatrix}\ . \tag{30}$$

Because the poles are coordinate singularities, the stability of the stationary point $P_1$, and of $P_2$ when the transverse momentum $I_2=0$ will be studied separately.

From Theorem 4 case *(ii)*, a stationary point $T$ can only occur for angles $\Phi_3$ for which $\cos(\sqrt{11}\Phi_3)=(-1)^m$, at an intersection of the graphs of $F_1(M_3)$ and $F_2(M_3)$. At such points $T$, the matrix forming the RHS of (30) takes the form

$$\begin{pmatrix}0 & \partial_{M_3}(F_1\pm F_2)\\ \pm 66W_{111\overline{23}}I^{3/2}(k^{(1)})\sqrt{I(k^{(2)})I(k^{(3)})} & 0\end{pmatrix}\ . \tag{31}$$

In case $m$ is odd, then $T$ is elliptic (and therefore stable) if $66W_{111\overline{23}}\partial_{M_3}(F_1 - F_2) > 0$, and $T$ is hyperbolic in tangent directions to $S(a)$ if this quantity takes the other sign. This is to say that the criterion of stability is for $\partial_{M_3}F_1 > \partial_{M_3}F_2$, which holds if the graph of $F_1(M_3)$ crosses that of $F_2(M_3)$ from below to above, as $M_3$ increases.

In case $m$ is even, the analogous criterion holds; a stationary point $T$ will be stable if $-66W_{111\overline{23}}\partial_{M_3}(F_1 + F_2) > 0$, corresponding to the graph of $-F_1(M_3)$ crossing that of $F_2(M_3)$ from below to above, and unstable otherwise.

Without returning to complex symplectic coordinates, it is still possible to understand the nature of the poles $P_1 = (\Phi_3, M_3^-)$, and $P_2 = (\Phi_3, M_3^+)$ when $I(k^{(2)}) = I(k^{(3)})$. In particular, if $\partial_{\Phi_3}H_{BKS}(P_j) = 0$ and $\partial_{M_3}H_{BKS}(P_j)$ does not change sign, then $P_j$ is an elliptic stationary point. This is always the case for $P_1$. However the component $\partial_{M_3}H_{BKS}(\Phi_3, M_3^+)$ could change sign for some values of the momenta $I_1$, $I_2$, vanishing at angles $\Phi_3 = \Phi_3^\pm$. Then $P_2$ is hyperbolic in a tangent direction to $S(a)$ and $\Phi_3^\pm$ are the angles at which the stable and unstable manifolds approach it.

**The sequence of bifurcation events:** The information contained in the discussion above allows us to sketch a sequence of bifurcation events and changes of stability that occur for the BKS model (23). Similar sequences of bifurcation events occur for any three degree of freedom system whose Hamiltonian in Birkhoff normal form is expressed as in (18), where $V_{\ell\bar{\ell}j\bar{j}}$ and $W \neq 0$ are arbitrary.

There are two families of traveling wave solutions of (23) which exist for all values of the parameters. In the phase space $S(a)$ one occurs at the pole $T_1 = P_1$, corresponding to a traveling water wave $\eta_H(x - tc)$ which has the form of a hexagonal pattern. Inspecting the equation (28), a second solution $T_2$ occurs on the line $\sqrt{11}\Phi_3 = \pi$, for some value of $M_3 = M_3^{(2)}$, which varies depending upon the momentum parameters $(I_1, I_2)$. When $I_2(M) = 0$, then $T_2 = P_2$ occurs at the second pole, and as a traveling water wave solution it corresponds to the Stokes traveling wave train $\eta_S(x - tc)$. In addition to these, there is a sequence of secondary bifurcations, with concombinant changes of stability, which are described below. We will parameterize the families of solutions with the two momenta $(I_1, I_2)$.

**(1)** For values of the momentum $I_1 > 0$ which are small, $F_1 = O(I_1)$ while $F_2 = O(I_1^{3/2})$, and the constants $C_{BKS}(1,3)$ (which are symplectic invariants) satisfy $C_{BKS}(1,3) < 0$. An elementary analysis of the phase plane $S(a)$ (which is best done in a sketch by hand) implies that the only stationary points of system (23) on $S(a)$ occur at $T_1 = P_1$ and $T_2$ on the line $\sqrt{11}\Phi_3 = \pi$. According to the stability criterion of the previous subsection, both of these stationary points are elliptic.

**(2)** Fixing $I_2(M) = 0$, the pole $T_2 = P_2$ remains stable as $I_1(M)$ increases, as long as $F_1(M_3^+) < F_2(M_3^+)$. Since $F_2(M_3^+) < 0$ there is a point $I_1 = b_1$ at which $F_1(M_3^+) = F_2(M_3^+)$, beyond which the stationary point $T_2 = P_2$ looses stability to a new stationary point $T_3 = (M_3^{(3)}, \pi)$ which moves away from $P_2$ and toward $P_1$. Referring to the stability criteria for $m$ odd, the new solution $T_3$ is stable, while $P_2 = T_2$ now possesses a stable and an unstable manifold in $S(a)$. This new solution corresponds to a genuinely three-dimensional traveling water wave $\eta_C(x - tc)$, having the form of a doubly periodic crescent shaped wave pattern. Simultaneously, an orbit in $S(a)$ homoclinic to $T_2$ is formed which surrounds the new elliptic stationary point $T_3$. For $I_1$ exceeding $b_1$ by only a little, the homoclinic orbit passes very close to $T_3$.

On phase planes $S(a)$ with $I_2 \neq 0$, the elliptic stationary point $T_2$ moves along the line $\Phi_3 = \pi$ with monotonically decreasing $M_3^{(3)}$.

**(3)** In parallel with the event (2), for $m$ even one considers the line $\sqrt{11}\Phi_3 = 0$. For small $I_1 > 0$ the graphs of $-F_1(M_3)$ and $F_2(M_3)$ do not intersect, however at some point $I_1 = b_2$ the two graphs meet at a first value of $M_3^- < M_3^{(4)} < M_3^+$, subsequent to which a pair of stationary points $T_4, T_5$ is formed. The one (say, $T_4$) with the smaller value of $M_3$ is hyperbolic, and the other is elliptic, as per the above stability criterion. Depending upon the details of the function $F_1$ and its dependence upon the parameters $I_1, I_2$, either bifurcation $b_1$ or $b_2$ occurs before the other, but the local character of the two bifurcations are independent of this.

From the structure of the nonlinear terms, no further bifurcations occur for the system (23).

**Time reversal symmetry:** The discrete symmetry $(t, \eta, \xi) \mapsto (-t, \eta, -\xi)$ of the original system (5) is preserved by the BKS model Hamiltonian system with Hamiltonian $H_{BKS}$. Any solution which exhibits a crescent shaped element facing forwards is therefore paired with another solution of the system which has crescent shaped elements facing backwards. In fact with the BKS model the crescent shaped traveling wave solutions $\eta_C(x - tc)$ have both forward and backward facing crescent shaped elements, and the time reversal transformation maps this solution to itself modulo a spatial translation. This property is the one shortcoming of the model in its ability to describe the three dimensional instability of the Stokes wave train $\eta_S(x - tc)$ and the tendency to form strongly three dimensional crescent shaped and forward facing periodic traveling wave solutions.

The time reversal transformation which takes the model (23) to itself and which fixes the traveling wave solution $T_2$ involves time reflection and spatial translation. Any traveling wave solution of the model with Hamiltonian (18) has a free surface profile given in the form

$$\eta_T(x) = a_1 \cos(k^{(1)} \cdot x + \theta_1) + a_2 \cos(k^{(2)} \cdot x + \theta_2) + a_3 \cos(k^{(3)} \cdot x + \theta_3). \quad (32)$$

**Proposition 6** *There is a translation $x \mapsto x + \alpha$ such that the phases can be set to $\theta_2 = 0 = \theta_3$. Suppose that $\theta_1 = \pi/3$, and consider the reflection $x_1 \mapsto -x_1$. There is a second translation $x \mapsto x + \beta$ such that the function $\eta_T(x)$ is a fixed point of the composition.*

**Proof** Set $x = (x_1, x_2) = (y_1 + \alpha_1, y_2 + \alpha_2)$, and consider its effect on the two phases $\theta_2, \theta_3$. If $\alpha$ satisfies

$$\begin{pmatrix} \frac{3}{2} & k_2^{(2)} \\ \frac{3}{2} & -k_2^{(2)} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix}$$

then the goal of the first statement is achieved. We note that the determinant of the above transformation is $-3k_2^{(2)} \neq 0$. After this first transformation, perform the reflection $(x_1, x_2) \mapsto (-x_1, x_2)$. The phases in the expression (32) become $(-k_1^{(1)} x_1 + \theta_1, -3/2k_1^{(1)} x_1 + k_2^{(2)} x_2, -3/2k_1^{(1)} x_1 - k_2^{(2)} x_2)$. Setting $(x_1, x_2) = (y_1 + 2(\theta_1/k_1^{(1)}), y_2 - (\pi/k_2^{(2)}))$, and noting that (32) has been written in cosine series, the phases become $(k_1^{(1)} y_1 + \theta_1, 3/2k_1^{(1)} y_1 - k_2^{(2)} y_2 + 3\theta_1 + \pi, 3/2k_1^{(1)} y_1 + k_2^{(2)} y_2 + 3\theta_1 - \pi)$. For a traveling wave solution for which $\theta_1$ is any odd multiple of $\pi/3$ and $a_2 = a_3$, this recovers expression (32). In the case of the traveling wave solutions $T_2$, which are the ones which exhibit the strongly crescent shaped features reminiscent of the experiments of [24][25] and of [2], both $\theta_1 = \pi/3$ and $a_2 = a_3$. $\square$

## 5. A New Deep Water Resonant Model:

The BKS model Hamiltonian involves three Fourier modes, and its character depends upon one fifth order resonant interaction between them. The derivation of the model involves an assumption that these three modes are the only ones of principal importance to the instability of the Stokes wave train to a crescent shaped wave, and in the model the amplitudes of all other Fourier modes are set to zero. However in the deep water problem, with $h = +\infty$, then $\omega^2(k) = g|k|$ and there is another prominent fifth order resonance relation that might be suspected of playing a rôle. Specifically, we include the Fourier mode with wave number $k^{(4)} = 4k^{(1)}$, which adds another resonant quintet

$$3\omega(k^{(1)}) - \omega(-k^{(1)}) - \omega(k^{(4)}) = 0 \quad (33)$$

to the reduced system. The additional resonant interaction is arguably quite important to the dynamics, especially in considering water wave solutions close to the Stokes wave train, as they involve a principal component of the solution in the mode of wave number $k^{(1)}$. The model taking this into account is derived under the same considerations at that of the BKS model, however involving the four Fourier modes with wave numbers $k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)}$. The water wave Hamiltonian is transformed to Birkhoff normel form to fifth order, and then restricted to the subspace $\{z \ : \ z(k) = 0 \text{ for all } k \neq k^{(j)}, j = 1 \dots 4\}$, and it takes the form

$$
\begin{aligned}
H_R = & \sum_{\ell=1}^{4} \omega(k^{(j)})|z(k^{(j)})|^2 \\
& + \tfrac{1}{2} \textstyle\sum_{\ell=1}^{4} V_{\ell\ell\overline{\ell\ell}} |z(k^{(\ell)})|^4 + 3 \textstyle\sum_{\ell=1}^{4} \sum_{j\neq\ell} V_{\ell\ell\overline{jj}} |z(k^{(j)})|^2 |z(k^{(\ell)})|^2 \\
& + 6W_{111\overline{23}} \ \mathrm{re} \ (z(k^{(1)})^3 \overline{z(k^{(2)}) z(k^{(3)})}) \\
& + W_{111\overline{14}} \ \mathrm{re} \ z(k^{(1)})^3 \overline{z(k^{(1)}) z(k^{(4)})} \ .
\end{aligned} \tag{34}
$$

The two components of the horizontal momentum are

$$
\begin{aligned}
I_1 &= I(k^{(1)}) + \tfrac{3}{2}(I(k^{(2)}) + I(k^{(3)})) + 2I(k^{(4)}) \ , \\
I_2 &= \tfrac{3}{2}(I(k^{(2)}) - I(k^{(3)})) \ ,
\end{aligned} \tag{35}
$$

where $|z(k^{(j)})|^2 = I(k^{(j)})$, and these are conserved quantities under the time evolution

$$
\partial_t z(k^{(j)}) = i \partial_{\overline{z}(k^{(j)})} H_R(z) \ . \tag{36}
$$

To put this system into rotating coordinates, perform the canonical transformation

$$
\Psi = R\theta \ , \qquad N = (R^T)^{-1} I \ , \tag{37}
$$

where $R$ is given by

$$
R = \begin{pmatrix} \sqrt{\frac{2}{19}} & \frac{3}{2}\sqrt{\frac{2}{19}} & \frac{3}{2}\sqrt{\frac{2}{19}} & 2\sqrt{\frac{2}{19}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{3}{\sqrt{11}} & -\frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}} & 0 \\ \frac{2}{\sqrt{5}} & 0 & 0 & \frac{-1}{\sqrt{5}} \end{pmatrix} \ , \tag{38}
$$

then $\sqrt{11}\Psi_3 = 3\theta(k^{(1)}) - \theta(k^{(2)}) - \theta(k^{(3)})$ and $\sqrt{5}\Psi_3 = 2\theta(k^{(1)}) - \theta(k^{(4)})$. The Hamiltonian $H_R$ is transformed to

$$
\begin{aligned}
H_R = & \sum_{\ell=1}^{4} \omega(k^{(\ell)}) I(k^{(\ell)}) \\
& + \frac{1}{2} \sum_{\ell=1}^{4} V_{\ell\ell\overline{\ell\ell}} I(k^{(\ell)})^2 + 3 \sum_{\ell=1}^{4} \sum_{j \neq \ell} V_{\ell\bar{\ell}j\bar{j}} I(k^{(j)}) I(k^{(\ell)}) \\
& + 6 W_{111\overline{23}} I(k^{(1)})^{3/2} \sqrt{I(k^{(2)}) I(k^{(3)})} \cos(\sqrt{11}\Psi_3) \\
& + W_{111\overline{14}} I(k^{(1)})^2 \sqrt{I(k^{(4)})} \cos(\sqrt{5}\Psi_4) \ .
\end{aligned} \tag{39}
$$

The action variables $I(k^{(j)})$ are to be considered as functions of $N$. We observe that $H_R$ is independent of the two angles $\Psi_1, \Psi_2$, corresponding to the two integrals of motion $I_1, I_2$, however the Hamiltonian is not integrable in any obvious way. As with the BKS model, the reduced phase space is given by $S(a) = \{(N, \Phi) \ : \ I_1 = a_1, I_2 = a_2\}$, which is four dimensional. Traveling wave solutions with momenta $I_j = a_j, j = 1, 2$ correspond to stationary points of $H_R$ on $S(a)$, with the components of its phase velocity of a solution $T$ given by

$$
\partial_{N_1} H_R(T) = \sqrt{\tfrac{19}{2}} c_1 \ , \quad \partial_{N_2} H_R(T) = \tfrac{\sqrt{2}}{3} c_2 \ . \tag{40}
$$

**Proposition 7** *On the reduced phase space $S(a)$, stationary points of $H_R$ occur either for*

*(i)* $\sqrt{11}\Psi_3 = 3\theta(k^{(1)}) - \theta(k^{(2)}) - \theta(k^{(3)}) = m_1 \pi$

*(ii)* $\sqrt{5}\Psi_4 = 2\theta(k^{(1)}) - \theta(k^{(4)}) = m_2 \pi$,

*for some integers $m_1, m_2$, or else for $I(k^{(1)})(N) = 0$. The latter case corresponds to a traveling wave solution $\eta_H(x - tc)$ exhibiting a hexagonal spatial pattern. If the transverse momentum $I_2 = 0$, then there is a third possibility, in which the set $\{I(k^{(2)}) = 0 = I(k^{(3)})\}$ contains a stationary point of $H_R$ on $S(a)$. This corresponds to the Stokes wave train solution $\eta_S(x - tc)$.*

**Proof** Stationary points which are not at singularities of the symplectic polar coordinates must satisfy

$$
\begin{aligned}
0 = \partial_{\Psi_3} H_R = & -6\sqrt{11} W_{111\overline{23}} I(k^{(1)})^{3/2} \sqrt{I(k^{(2)}) I(k^{(3)})} \sin(\sqrt{11}\Psi_3) \\
0 = \partial_{\Psi_4} H_R = & \sqrt{5} W_{111\overline{14}} I(k^{(1)})^2 \sqrt{I(k^{(4)})} \sin(\sqrt{5}\Psi_4) \ .
\end{aligned} \tag{41}
$$

The effect is that (ii) holds, which for each $m_2 \in \mathbb{Z}$ fixes $\Psi_4$ as a function of $\Psi_3$. Furthermore (i) holds, setting $\Psi_3$ to discrete values similar to the critical points of the BKS Hamiltonian. Upon inspection, $I(k^{(1)}) = 0$ is a stationary point of $H_R$ on $S(a)$ at the pole $P_1$, but $I(k^{(4)}) = 0$ only occurs as a coordinate singularity. $\square$

Critical points of (39) which do not occur at one of the poles will satisfy (41) as well as

$$
\begin{aligned}
0 =& \partial_{N_3} H_R \\
=& \partial_{N_3} B(I(N)) + \tfrac{6}{\sqrt{11}} W_{111\overline{23}} \Big( \tfrac{9}{2} \sqrt{I(k^{(1)}) I(k^{(2)}) I(k^{(3)})} \\
& -\tfrac{1}{2} I(k^{(1)})^{3/2} \Big( \sqrt{\tfrac{I(k^{(3)})}{I(k^{(2)})}} + \tfrac{I(k^{(2)})}{I(k^{(3)})} \Big) \Big) \cos(\sqrt{11}\Psi_3) \\
& + \tfrac{6}{\sqrt{11}} W_{111\overline{14}} I(k^{(1)}) \sqrt{I(k^{(4)})} \cos(\sqrt{5}\Psi_4) \ ,
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
0 =& \partial_{N_4} H_R \\
=& \partial_{N_4} B(I(N)) + \tfrac{18}{\sqrt{5}} W_{111\overline{23}} \sqrt{I(k^{(1)}) I(k^{(2)}) I(k^{(3)})} \cos(\sqrt{11}\Psi_3) \\
& + \tfrac{1}{\sqrt{5}} W_{111\overline{14}} \Big( 4 I(k^{(1)}) \sqrt{I(k^{(4)})} - \tfrac{1}{2} I(k^{(1)})^2 \sqrt{I(k^{(4)})}^{-1} \Big) \cos(\sqrt{5}\Psi_4) \ ,
\end{aligned}
\tag{43}
$$

where $B(I(N))$ is the quadratic form in the action variables which appears in (39). The coefficients of the Hamiltonian $H_R$ in (39) are taken to satisfy the structure conditions $W_{111\overline{23}} > 0$, $W_{111\overline{14}} > 0$, $C_R(1,3) = 3V_{1\bar{1}1\bar{1}} - V_{1\bar{1}2\bar{2}} - V_{1\bar{1}3\bar{3}} < 0$, and $C_R(1,4) = \sqrt{22} V_{1\bar{1}1\bar{1}} - V_{1\bar{1}4\bar{4}} < 0$. These are consistent with the BKS model and [3][6].

Given a critical point $T$ of (39) on $S(a)$ which is not at a pole, and given the result of Proposition 7 a stability criterion for the system is straightforward to work out.

**Proposition 8** *Suppose that a stationary point $T$ of the system with Hamiltonian (39) on $S(a)$ is not one of the poles. Then $T$ is stable if both*

$$
\partial^2_{N_3} H_R \partial^2_{\Psi_3} H_R + \partial^2_{N_4} H_R \partial^2_{\Psi_4} H_R > 0 \ , \tag{44}
$$

$$
\partial^2_{\Psi_3} H_R \partial^2_{\Psi_4} H_R \left( \partial^2_{N_3} H_R \partial^2_{N_4} H_R - (\partial_{N_3}\partial_{N_4} H_R)^2 \right) > 0 \ . \tag{45}
$$

**The sequence of bifurcation events:** There is now a clear sequence of bifurcation events for traveling wave solutions of system given by (39). This is very closely related to the squence of bifurcation events for the BKS system.

**(1)** For small $x_1$-momentum $I_1$, each phase plane $S(a)$ contains only two critical points of $H_R$; one which is at the pole $T_1 = \{I(k^{(1)}) = 0\}$, and the other $T_2$ close to the pole $\{I(k^{(3)}) = 0\}$ (we are again normalizing so that $0 < I(k^{(3)}) \le I(k^{(2)})$. When $I(k^{(3)}) = I(k^{(2)})$ then this critical point $T_2$ occurs at the pole $\{I(k^{(2)}) = I(k^{(3)}) = 0\}$ itself.

**(2)** At some bifurcation point $I_1 = c_1 > b_1$, on the phase plane $S(a)$ such that $I(k^{(2)}) = I(k^{(3)})$, the stationary point $T_2$ loses stability in a Hamiltonian saddle-node bifurcation. A new critical point $T_3$ emerges on the phase plane $S(a)$ for which $\sqrt{11}\Psi_3 = \pi$ and $I(k^{(2)}) = I(k^{(3)}) > 0$. These solutions $\eta_C(x-tc)$ have three dimensional structure. In particular they feature the phase shift $\theta_1 = \pi/3$ giving them the characteristic feature of crescent shaped water waves. According to the stability criterion of Proposition 8, the solution $T_3$ is stable.

**(3)** At a separate bifurcation point $I_1 = c_2 > b_2$, a pair of stationary points $T_4$ and $T_5$ is created on the set $\{\sqrt{11}\Psi_3 = 0\}$. At these points all action variables $I(k^{(\ell)}), \ell = 1 \ldots 4$ are nonzero. One of these critical points is stable, while the other is very likely to be unstable (it depends in general on the parameters).

Further bifurcations can occur for larger values of the momentum $I_1$, but these are less relevant to the water waves problem as they take place further from the weakly nonlinear regime under which this model is derived, and we will not describe the resulting stationary points here.

**Time reversal symmetry:** The time reversal symmetry $(t, \eta, \xi) \mapsto (-t, \eta, -\xi)$ leaves the $R$-system invariant. Similarly to the BKS model, time reflection composed with a specific translation maps the candidate crescent shaped solutions $\eta_C(x-tc)$ to themselves. It remains an open problem whether a Hamiltonian model of water waves will exhibit solutions which satisfy all three of the criteria for nonlinear crescent shaped solutions that are specified in [22], which are closer to the desired unidirectional crescent shaped waveforms of the wave tank experiments of [24][25].

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