THREE-DIMENSIONAL SOLITARY GRAVITY-CAPILLARY WATER WAVES

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The existence of solitary-wave solutions to the three-dimensional water-wave problem with strong surface-tension effects is predicted by the KP-I model equation. The term *solitary wave* describes any solution which has a pulse-like profile in its direction of propagation, and the KP-I equation admits explicit solutions for three different types of solitary wave. A *line solitary wave* is spatially homogeneous in the direction transverse to its direction of propagation, while a *periodically modulated solitary wave* is periodic in the transverse direction. A *fully localised solitary wave* on the other hand decays to zero in all spatial directions. In this article we outline mathematical results which confirm the existence of all three types of solitary wave for the full gravity-capillary water-wave problem in its usual formulation as a free-boundary problem for the Euler equations.

1. Introduction

The classical *three-dimensional gravity-capillary water wave problem* concerns the irrotational flow of a perfect fluid of unit density subject to the forces of gravity and surface tension. The fluid motion is described by the Euler equations in a domain bounded below by a rigid horizontal bottom $\{Y = 0\}$ and above by a free surface which is described as a graph $\{Y = h + \rho(x, z, t)\}$, where h denotes the depth of the water in its undisturbed state and the function ρ depends upon the two horizontal spatial directions x, z and time t. *Steady waves* are water waves which are uniformly translating in a distinguished horizontal direction without change of shape; without loss of generality we assume that the waves propagate in the x-direction with speed c and continue to write x as an abbreviation for $x - ct$. In terms of an Eulerian velocity potential $\phi(x, Y, z, t)$ the mathematical problem for steady waves is to solve the equations

$$\phi_{xx} + \phi_{YY} + \phi_{zz} = 0 \qquad 0 < Y < 1 + \rho, \tag{1}$$

$$\phi_Y = 0 \qquad \text{on } Y = 0, \tag{2}$$

$$\phi_Y = \rho_x \phi_x + \rho_z \phi_z - \rho_x \qquad \text{on } Y = 1 + \rho \tag{3}$$

and

$$\begin{aligned} &-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_Y^2 + \phi_z^2) + \alpha\rho \\ &- \beta \left[\frac{\rho_x}{\sqrt{1 + \rho_x^2 + \rho_z^2}} \right]_x - \beta \left[\frac{\rho_z}{\sqrt{1 + \rho_x^2 + \rho_z^2}} \right]_z = 0 \quad \text{on } Y = 1 + \rho \end{aligned} \tag{4}$$

(see Stoker[1]), in which we have introduced dimensionless variables. The equations involve two physical parameters $\alpha := gh/c^2$ and $\beta := \sigma/hc^2$, where g and σ are respectively the acceleration due to gravity and the coefficient of surface tension.

The steady water-wave problem (1)–(4) is a free boundary-value problem with nonlinear boundary conditions, and in this respect its solution poses considerable mathematical difficulties. At a formal level these difficulties may be overcome by replacing the above equations by a simpler model equation based upon certain approximations. One of the more widely used model equations is the KP-I equation

$$\partial_{xx} \left(u_{xx} - u - \frac{3}{2}u^2 \right) - u_{zz} = 0, \tag{5}$$

in which u depends upon two unbounded spatial directions x and z. This equation was derived formally by Kadomtsev and Petviashvili[2] as a long-wave approximation for solutions of the steady gravity-capillary water-wave problem (1)–(4) in which

$$\beta > 1/3, \qquad \alpha = 1 + \epsilon, \quad 0 < \epsilon \ll 1; \tag{6}$$

the variable u is supposed to approximate the free surface of the water via the formula

$$\rho(x, z) = \epsilon\, u \left(\frac{\epsilon^{1/2} x}{2(\beta - 1/3)^{1/2}}, \epsilon z \right) + O(\epsilon^2).$$

The KP-I equation (5) admits a family of explicit solutions $\{u^\delta\}_{\delta \in [0,1)}$ which are given by the formulae

$$u^\delta(x, z) = -\frac{4(1 - \delta^2)}{4 - \delta^2} \frac{1 - \delta \cosh(a^\delta x) \cos(\omega^\delta z)}{(\cosh(a^\delta x) - \delta \cos(\omega^\delta z))^2},$$

where

$$a^\delta = \sqrt{\frac{1-\delta^2}{4-\delta^2}}, \qquad \omega^\delta = \frac{\sqrt{3(1-\delta^2)}}{4-\delta^2}$$

(see Tajiri and Murakami[3]). These waves are *periodically modulated solitary waves* which decay exponentially to zero as $x \to \pm\infty$ and are periodic with frequency ω^δ in the transverse spatial direction z. In the special case $\delta = 0$ we obtain the solution

$$u^0(x) = -\mathrm{sech}^2\left(\frac{x}{2}\right),$$

which in view of its spatial homogeneity with respect to z is called a *line solitary-wave solution* of the KP-I equation. One can also take the limit as $\delta \to 1$ in the above formula; the limiting function u^1 is given by

$$u^1(x,z) = -8\frac{3-x^2+z^2}{(3+x^2+z^2)^2}$$

and defines a *fully localised solitary-wave solution* of the KP-I equation[4] which decays algebraically to zero in both spatial directions. The three types of solitary wave are sketched in Fig. 1.

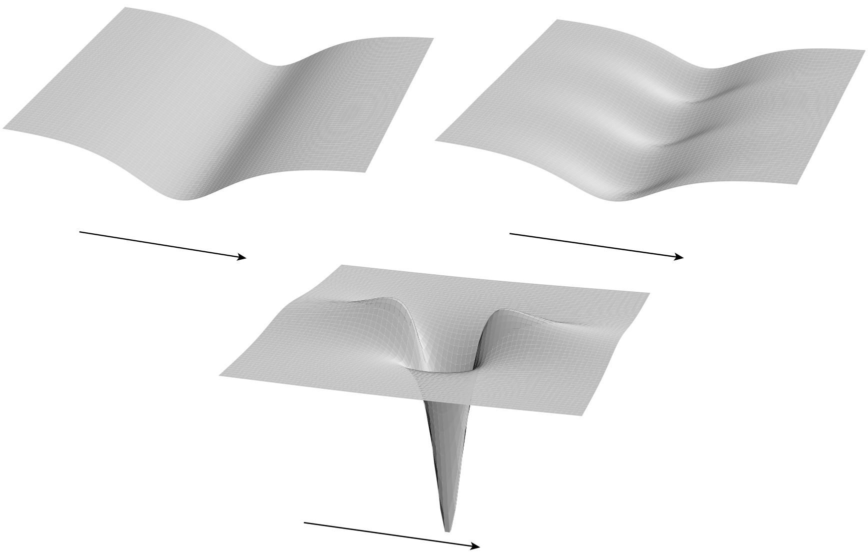

Figure 1. Clockwise from top left: line, periodically modulated and fully localised solitary waves; the arrow shows the direction of wave propagation.

The above discussion shows that, at a formal level, the KP-I equation predicts the existence of three kinds of solitary-wave solution to the water-wave problem (1)–(4) in the parameter regime (6). This prediction has inspired intensive research into the solution set of the water-wave problem itself, and rigorous theories have recently become available which confirm the prediction: the steady water-wave problem admits line, periodically modulated and fully localised solitary-wave solutions which are qualitatively similar to those of the KP-I equation; in particular they are all waves of depression. The elements of these theories, which are due to Kirchgässner[5], Groves, Haragus and Sun[6] and Groves and Sun[7], are reviewed below.

2. A variational principle

The key to the existence theories for solitary water waves is the observation that equations (1)–(4) in the parameter regime (6) follow from the formal variational principle

$$\delta\left\{ \iint \left(\int_0^{1+\rho} (-\phi_x + \frac{1}{2}(\phi_x^2 + \phi_Y^2 + \phi_z^2))\,\mathrm{d}Y \right.\right.$$
$$\left.\left. + \frac{1}{2}(1+\epsilon)\rho^2 + \beta(\sqrt{1+\rho_x^2+\rho_z^2} - 1) \right) \mathrm{d}(x,z) \right\} = 0,$$

where the variation is taken in (ρ, ϕ)[8]. A more satisfactory version of this variational principle is obtained using the change of variable

$$Y = y(1+\rho(x,z)), \qquad \phi(x,Y,z) = \Phi(x,y,z), \tag{7}$$

which transforms the variable fluid domain into the fixed domain $\{0 < y < 1\}$. One obtains the new variational principle

$$\delta\mathcal{F} = 0, \qquad \mathcal{F} = \iint F(\rho,\Phi)\,\mathrm{d}(x,z), \tag{8}$$

in which

$$F(\rho,\Phi) = \frac{1}{2}(1+\epsilon)\rho^2 + \beta[\sqrt{1+\rho_x^2+\rho_z^2} - 1] + \rho_x \Phi|_{y=1}$$
$$+ \int_0^1 \left(\frac{1}{2}\left[\Phi_x - \frac{y\rho_x\Phi_y}{1+\rho}\right]^2 + \frac{\Phi_y^2}{2(1+\rho)^2} + \frac{1}{2}\left[\Phi_z - \frac{y\rho_z\Phi_y}{1+\rho}\right]^2 \right)(1+\rho)\,\mathrm{d}y.$$

A variational principle for a physical problem can be exploited in two ways. Firstly, it may be possible to use the direct methods of the calculus of variations to find critical points of the variational functional and hence solutions of the problem. This approach is applied to the variational principle (8) in the existence theory for fully localised solitary waves (see Sec. 5 below). The second possibility is to regard the variational functional as an action functional and use classical theory to re-formulate the associated Euler-Lagrange equations as a Hamiltonian system.

With a slight abuse of notation, we write the variational principle (8) as

$$\delta \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} F(\rho, \Phi, \rho_x, \Phi_x) \,\mathrm{d}z \right) \mathrm{d}x = 0,$$

which takes the form of Hamilton's principle for an action functional in which x is the time-like variable, (ρ, Φ) are the coordinates and (ρ_x, Φ_x) the corresponding velocities. Following the classical theory, we take the Legendre transform and hence derive the (infinite-dimensional) Hamiltonian system

$$u_x = Lu + N(u), \tag{9}$$

where $u = (\rho, \Phi, \omega, \Psi)$, $\omega = \delta_{\rho_x}\mathcal{F}$, $\Psi = \delta_{\Phi_x}\mathcal{F}$ and δ denotes a variational derivative. A solution of this Hamiltonian system defines a steady water wave via the formula (7). Alternatively, we may write the variational principle as

$$\delta \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} F(\rho, \Phi, \rho_z, \Phi_z) \,\mathrm{d}x \right) \mathrm{d}z = 0,$$

so that z is now the time-like variable and (ρ_z, Φ_z) the velocities; the Legendre transform now yields the Hamiltonian system

$$u_z = Lu + N(u), \tag{10}$$

where $u = (\rho, \Phi, \omega, \Psi)$, $\omega = \delta_{\rho_z}\mathcal{F}$, $\Psi = \delta_{\Phi_z}\mathcal{F}$. In both cases the Legendre transform can be carried out explicitly, so that the relevant Hamiltonian systems can also be computed explicitly. An integration by parts with respect to the vertical coordinate y is required during this procedure, so that boundary conditions at $y = 0$ and $y = 1$ emerge, and the boundary condition at $y = 1$ is in fact nonlinear. This difficulty is overcome by a change of variable which converts the Hamiltonian systems into equivalent Hamiltonian systems with linear boundary conditions. Full details of all steps of the above procedure are given by Groves and Mielke[9] (with x as 'time') and Groves[10] (with z as 'time').

The device of using an unbounded spatial coordinate as a time-like variable in an evolutionary equation was introduced by Kirchgässner[11] and has become known as 'spatial dynamics'. The above spatial dynamics formulations (9) and (10) of the steady water-wave problem form the basis of the existence theories for respectively line and periodically modulated solitary waves and are discussed in Secs. 3 and 4 below. The Hamiltonian structure plays an important role in these theories, as does the fact that the systems are *reversible*: equations (9) and (10) are invariant under the transformations $x \mapsto -x$, $(\rho, \omega, \Phi, \Psi) = (\rho, -\omega, -\Phi, \Psi)$ and $z \mapsto -z$, $(\rho, \omega, \Phi, \Psi) \mapsto (\rho, -\omega, \Phi, -\Psi)$ respectively, a manifestation of the fact that our waves have no preferred direction of propagation.

3. Line solitary waves

Let us examine the spatial dynamics formulation

$$u_x = Lu + N(u) \tag{11}$$

of the steady water-wave problem in which x is the time-like variable. Recall that equation (11) represents Hamilton's equations for a reversible Hamiltonian system (M, Ω, H), and here we consider this evolutionary equation in a phase space $\mathcal{X}$ of functions which are z-independent, so that its solutions define *two-dimensional* water waves. In particular, homoclinic solutions (global solutions which decay to zero as $x \to \pm\infty$) correspond to line solitary waves.

The first step in the analysis of the above evolutionary equation is to examine the spectrum of its linear operator L. One finds that $\sigma(L)$ consists of isolated eigenvalues of finite multiplicity; a geometrically simple zero eigenvalue with a Jordan chain of length 2 is created in a *Hamiltonian* 0^2 *resonance* as ϵ is decreased to zero, while the rest of the spectrum is bounded away from the imaginary axis (Fig. 2).

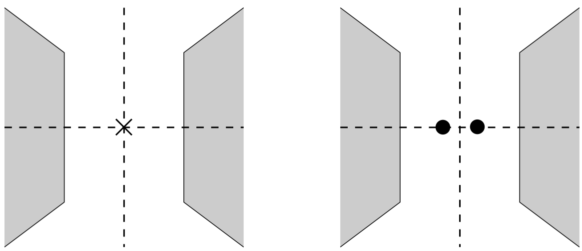

Figure 2. The spectrum of the linear operator L for zero (left) and small, non-negative values (right) of ϵ. A Hamiltonian 0^2 resonance takes place at the origin (dots and crosses denote eigenvalues which are geometrically simple and respectively algebraically simple and double) while the rest of the spectrum is contained in wedge-like regions (shaded) bounded away from the imaginary axis.

It is possible to find solutions of (11) by applying a centre-manifold reduction theorem[12,13]; one finds that this equation is locally equivalent to a Hamiltonian system with one degree of freedom representing the 'normal form' for the Hamiltonian 0^2 resonance. This method was first used by Kirchgässner[5] for the present problem (in a framework which did not include the Hamiltonian structure).

Theorem 3.1. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ denote the centre and hyperbolic subspaces of $\mathcal{X}$ determined by the operator L at $\epsilon = 0$, so that $\mathcal{X}_1$ is two-dimensional. There exist neighbourhoods Λ of 0 in $\mathbb{R}$ and U_1,U_2 of 0 in $\mathcal{D}(K) \cap \mathcal{X}_1$, $\mathcal{D}(K) \cap \mathcal{X}_2$ together with a reduction function $h \in U_1 \times \Lambda \to U_2$ which satisfies $h(0,0) = 0$, $\mathrm{d}_1 h[0,0] = 0$ and has the following properties. For each $\epsilon \in \Lambda$ the graph*

$$M_C^\epsilon = \{x_1 + h(x_1, \epsilon) \in U_1 \times U_2 : x_1 \in U_1\}$$

is a locally invariant two-dimensional manifold of (11), every small, bounded solution of (11) lies on M_C^ϵ, and M_C^ϵ is a symplectic submanifold of M. Moreover, the flow determined by the Hamiltonian system $(M_C^\epsilon, \tilde{\Omega}, \tilde{H})$, where the tilde denotes restriction to M_C^ϵ, coincides with the flow on M_C^ϵ determined by (M, Ω, H).

Hamilton's equations for $(M_C^\epsilon, \tilde{\Omega}, \tilde{H})$ are conveniently studied in terms of the scaled variables

$$Q = \epsilon^{-1}(\beta - 1/3)^{-1/2} q, \quad P = \epsilon^{-3/2} p, \quad X = \epsilon^{1/2}(\beta - 1/3)^{-1/2} x,$$

where (q, p) is a symplectic coordinate system for $\mathcal{X}_1$.[5,13] One finds that

$$Q_X = P + \mathcal{R}_1(Q, P, \mu), \tag{12}$$

$$P_X = Q + \frac{3}{2} Q^2 + \mathcal{R}_1(Q, P, \mu), \tag{13}$$

in which the remainder terms $\mathcal{R}_1$ and $\mathcal{R}_2$ are $O(\epsilon^{1/2})$ and respectively odd and even in their second arguments. It is a straightforward exercise to sketch the phase portrait of this dynamical system for $\epsilon = 0$ (see Fig. 3), and since the system is Hamiltonian and reversible, elementary transversality arguments show that its phase portrait is qualitatively the same for small, positive values of ϵ. The phase portrait reveals the existence of a reversible homoclinic solution which yields the desired line solitary wave, and tracing back the various changes of variables one arrives at the asymptotic formula

$$\rho(x) = -\epsilon \operatorname{sech}^2 \left(\frac{\epsilon^{1/2} x}{2(\beta - 1/3)^{1/2}} \right) + O(\epsilon^2)$$

for the corresponding free surface (which is symmetric in x).

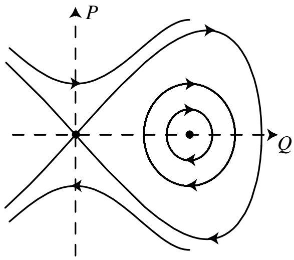

Figure 3. Phase portrait for the dynamical system (12), (13).

4. Periodically modulated solitary waves

We now turn to the spatial dynamics formulation

$$u_z = Lu + N(u) \tag{14}$$

of the steady water-wave problem in which z is the time-like variable. Recall that equation (11) is a reversible Hamiltonian system, and here we study it in a phase space $\mathcal{X}$ of symmetric functions which decay to zero as $x \to \pm\infty$, so that all its solutions are symmetric solitary waves. In particular, equilibrium and periodic solutions of (14) correspond to respectively line and periodically modulated solitary waves.

The line solitary wave found in Sec. 3 above defines an equilibrium solution $u^\star$ to (14), and we may use a translation

$$u(z) = u^\star + w(z) \tag{15}$$

to obtain the new Hamiltonian system

$$w_z = L^\star w + N^\star(w). \tag{16}$$

The following result by Groves, Haragus and Sun[6] concerns the spectrum of the linear operator $L^\star$.

Theorem 4.1. *The spectrum of $L^\star$ consists of two simple imaginary eigenvalues $\pm \mathrm{i}k_\epsilon$, where k_ϵ is $O(\epsilon)$, together with essential spectrum along the whole of the real axis (Fig. 4).*

At this point it is helpful to recall the classical Lyapunov centre theorem that a finite-dimensional Hamiltonian system with nonresonant imaginary eigenvalues $\pm \mathrm{i}k$ has a family of small-amplitude periodic solutions with frequency near k. Iooss[14] has recently established a result of this kind for

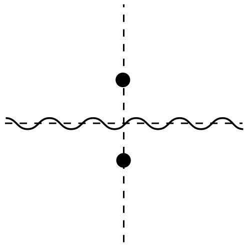

Figure 4. The spectrum of the linear operator $L^\star$ for small, non-negative values of ϵ: two simple imaginary eigenvalues of $O(\epsilon)$ together with essential spectrum along the real axis.

reversible systems in infinite-dimensional settings for which the nonresonance condition is violated at the origin due to the presence of essential spectrum.

Theorem 4.2. *Consider a quasilinear, reversible evolutionary equation*

$$u_t = Lu + N(u)$$

in the phase space $\mathcal{X}$. Suppose that the linear operator L has a pair $\pm\mathrm{i}k$ of simple imaginary eigenvalues, that 0 is contained in its essential spectrum, and that

(1) all nonzero integer multiples of $\pm\mathrm{i}k$ lie in the resolvent set of L;
(2) L satisfies the estimate $\|(L - \mathrm{i}\lambda I)^{-1}\| = O(\lambda^{-1})$ as $\lambda \to \pm\infty$;
(3) the range of the nonlinearity N lies in the range of L, so that the equation $Lv = -N(u)$ is solvable for each function u in the domain of N.

Under these conditions the above evolutionary equation admits a family of small-amplitude periodic solutions whose frequency is near k.

Groves, Haragus and Sun[6] verified that the conditions in Theorem 4.2 are satisfied by equation (16), which therefore has a family of periodic solutions with frequency of $O(\epsilon)$; a family of periodically modulated solitary water waves is obtained using formula (15).

5. Fully localised solitary waves

This existence theory is variational in nature and based upon the observation that each nonzero critical point of the functional

$$\mathcal{F} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\rho, \Phi)\,\mathrm{d}x\,\mathrm{d}z$$

corresponds to a fully localised solitary wave; here F is defined by the formula below (8) and we work in a space of functions which decay to zero as $|(x,z)| \to \infty$.

In applying the direct methods of the calculus of variations to a problem of this kind one is confronted with the lack of compactness associated with the unbounded domain $\{(x,z) \in \mathbb{R}^2\}$, and recently methods based upon the 'concentration-compactness' principle of Lions[15] have become available for dealing with such situations. However these methods have the key requirement that the corresponding Euler-Lagrange equations should be semilinear, or equivalently that the highest-order derivatives should appear in the quadratic part of the integrand of the associated variational functional. An examination of the formula for F shows that the highest-order derivatives appear at all orders (so that the Euler-Lagrange equations are quasilinear). This difficulty is overcome by a reduction theory which shows that all small-amplitude solutions may be obtained as solutions of a 'bifurcation equation' which is a *semilinear* partial differential equation; the bifurcation equation has a variational structure which is inherited from that of the full problem and is amenable to treatment using standard methods. The reduction method, which is similar in character to the variational Lyapunov-Schmidt reduction, is summarised below and described in detail by Groves and Sun[7].

We begin by adjusting the variational principle using the scaling

$$(x,y,z) \mapsto (\epsilon^{-1/2}x, y, \epsilon^{-1}z), \qquad (\rho,\Phi) \mapsto (\epsilon\rho, \epsilon^{1/2}\Phi)$$

relevant to the KP-I parameter regime and examining the Euler-Lagrange equation for ρ, namely

$$(1+\epsilon)\rho + \beta\epsilon\rho_{xx} + \beta\epsilon^2\rho_{zz} = \mathcal{N}_1(\rho,\Phi), \tag{17}$$

and for Φ, which takes the form of the boundary-value problem

$$-\epsilon\Phi_{xx} - \epsilon^2\Phi_{zz} - \Phi_{yy} = \mathcal{N}_2(\rho,\Phi), \quad 0<y<1, \tag{18}$$

$$\Phi_y = 0, \quad y=0, \tag{19}$$

$$\Phi_y + \epsilon\rho_x = \mathcal{N}_3(\rho,\Phi), \quad y=1, \tag{20}$$

where the symbol $\mathcal{N}_i(\rho,\Phi)$ denotes a nonlinear function of ρ and Φ.

Lemma 5.1. *Equation (17) is locally solvable and yields the functional relationship $\rho = \rho(\Phi)$.*

Inserting $\rho = \rho(\Phi)$ into the boundary-value problem (18)–(20), we arrive at a system of equations for the single variable Φ which follows from the reduced variational principle

$$\delta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\Phi)\,\mathrm{d}x\,\mathrm{d}z = 0, \qquad G(\Phi) := F(\rho(\Phi),\Phi),$$

in which the variation is taken with respect to Φ. Writing down the Euler-Lagrange equation and taking Fourier transforms, we obtain the equations

$$-\hat{\Phi}_{yy} + q^2\hat{\Phi} = \hat{\mathcal{N}_1}(\Phi), \qquad 0 < y < 1, \tag{21}$$

$$\hat{\Phi}_y = 0, \qquad y = 0, \tag{22}$$

$$\hat{\Phi}_y - \frac{\epsilon\mu^2\hat{\Phi}}{1+\epsilon+\beta q^2} = \hat{\mathcal{N}_2}(\Phi), \qquad y = 1, \tag{23}$$

where (μ, k) is the independent variable associated with the Fourier transform, the symbol $\mathcal{N}_i(\Phi)$ denotes a nonlinear function of Φ and $q^2 = \epsilon\mu^2 + \epsilon^2 k^2$. For later use let us note here that

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta G_{\mathrm{nl}}(\Phi)\delta\Phi\,\mathrm{d}x\,\mathrm{d}z\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\int_0^1 \mathcal{N}_1(\Phi)\delta\Phi\,\mathrm{d}y + \mathcal{N}_2(\Phi)\delta\Phi|_{y=1}\right\}\mathrm{d}x\,\mathrm{d}z,\end{aligned} \tag{24}$$

where $G_{\mathrm{nl}}(\Phi)$ is the function obtained from $G(\Phi)$ by subtracting its quadratic terms.

The boundary-value problem (21)–(23) can be re-formulated as the integral equation

$$\hat{\Phi} = -\int_0^1 G(y,\zeta)\hat{\mathcal{N}_1}(\Phi)\,\mathrm{d}\zeta - G(y,1)\hat{\mathcal{N}_2}(\Phi), \tag{25}$$

in which $G(y,\zeta)$ is the Green's function associated with the linear operator defined by its left-hand side, and we proceed by decomposing G into a leading-order singular part and a remainder term according to the formula

$$G(y,\zeta) = -\frac{1+\epsilon}{\epsilon^2 Q} + \epsilon^{-2}G_1(y,\zeta),$$

where

$$Q = k^2(1+\epsilon) + \mu^2 + (\beta - 1/3)\frac{q^4}{\epsilon^2} + c_0\frac{q^6}{\epsilon^2}$$

and $c_0 = \beta/2 - 2(1+\epsilon)/15$. Consider the equations

$$\hat{\Phi}_1 = \frac{1+\epsilon}{\epsilon^2 Q}\left(\int_0^1 \hat{\mathcal{N}_1}(\Phi_1+\Phi_2)\,\mathrm{d}\zeta + \hat{\mathcal{N}_2}(\Phi_1+\Phi_2)\right), \tag{26}$$

$$\hat{\Phi}_2 = -\int_0^1 \epsilon^{-2}G_1(y,\zeta)\hat{\mathcal{N}_1}(\Phi_1+\Phi_2)\,\mathrm{d}\zeta - \epsilon^{-2}G_1(y,1)\hat{\mathcal{N}_2}(\Phi_1+\Phi_2), \tag{27}$$

where $\Phi_1(x,z)$, $\Phi_2(x,y,z)$. Clearly any solution (Φ_1,Φ_2) of this pair of equations yields a solution $\Phi=\Phi_1+\Phi_2$ of (25), and conversely any solution Φ of (25) can be decomposed into a sum $\Phi=\Phi_1+\Phi_2$, where (Φ_1,Φ_2) solve (26), (27) (the functions Φ_1 and Φ_2 are calculated from the formulae obtained by replacing $\Phi_1+\Phi_2$ by Φ on the right-hand sides of (26), (27)). Equation (25) is therefore equivalent to (26), (27).

Observe that (26) and (27) can be written as respectively

$$\begin{aligned}&\frac{\epsilon^2}{1+\epsilon}\Big(-c_0(\epsilon^4\partial_z^6+3\epsilon^3\partial_x^2\partial_z^4+3\epsilon^2\partial_x^4\partial_z^2+\epsilon\partial_x^6)\Phi_1\\&\qquad+(\beta-1/3)(\epsilon^2\partial_z^4+2\epsilon\partial_x^2\partial_z^2+\partial_x^4)\Phi_1-(1+\epsilon)\partial_z^2\Phi_1-\partial_x^2\Phi_1\Big)\\&=\left(\int_0^1\mathcal{N}_1(\Phi_1+\Phi_2)\,\mathrm{d}\zeta+\mathcal{N}_2(\Phi_1+\Phi_2)\right)\end{aligned}\tag{28}$$

and

$$\begin{aligned}&-\hat{\Phi}_{2yy}+q^2\hat{\Phi}_2+\frac{(1+\epsilon)q^2}{\epsilon^2QR}\left(\int_0^1 q^2\hat{\Phi}_2\,\mathrm{d}y-\frac{\epsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\epsilon+\beta q^2}\right)\\&\qquad=\hat{\mathcal{N}}_1(\Phi_1+\Phi_2),\qquad\qquad 0<y<1,\end{aligned}\tag{29}$$

$$\hat{\Phi}_{2y}=0,\qquad\qquad y=0,\tag{30}$$

$$\begin{aligned}&\hat{\Phi}_{2y}-\frac{\epsilon\mu^2\hat{\Phi}_2}{1+\epsilon+\beta q^2}-\frac{(1+\epsilon)\epsilon\mu^2}{\epsilon^2Q(1+\epsilon+\beta q^2)R}\left(\int_0^1 q^2\hat{\Phi}_2\,\mathrm{d}y-\frac{\epsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\epsilon+\beta q^2}\right)\\&\qquad=\hat{\mathcal{N}}_2(\Phi_1+\Phi_2),\qquad\qquad y=1,\end{aligned}\tag{31}$$

where

$$R=1-\frac{(1+\epsilon)q^2}{\epsilon^2Q}+\frac{(1+\epsilon)\epsilon\mu^2}{\epsilon^2Q(1+\epsilon+\beta q^2)}.$$

The left-hand side of (28) defines a formally self-adjoint operator acting upon $\Phi_1(x,z)$ which is associated with the quadratic form

$$\begin{aligned}&Q_1(\Phi_1)=\\&\quad\frac{\epsilon^2}{2(1+\epsilon)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big\{c_0(\epsilon\Phi_{1xxx}^2+3\epsilon^2\Phi_{1xxz}^2+3\epsilon^3\Phi_{1xzz}+\epsilon^4\Phi_{1zzz}^2)\\&\qquad\qquad+(\beta-1/3)(\Phi_{1xx}^2+2\epsilon\Phi_{1xz}^2+\epsilon^2\Phi_{1zz}^2)\\&\qquad\qquad+\Phi_{1x}^2+(1+\epsilon)\Phi_{1z}^2\Big\}\,\mathrm{d}x\,\mathrm{d}z,\end{aligned}$$

and similarly the left-hand side of the boundary-value problem (29)–(31) defines a formally self-adjoint operator acting upon $\Phi_2(x,y,z)$ which is associated with the quadratic form

$$Q_2(\Phi_2) = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_0^1 (|\hat{\Phi}_{2y}| + q^2|\hat{\Phi}_2|)\,\mathrm{d}y - \frac{\epsilon\mu^2|\hat{\Phi}_2|_{y=1}|^2}{1+\epsilon+\beta q^2}\right.$$
$$\left. + \frac{1+\epsilon}{\epsilon^2 QR}\left|q^2\int_0^1 \hat{\Phi}_2\,\mathrm{d}y - \frac{\epsilon\mu^2\hat{\Phi}_2|_{y=1}}{1+\epsilon+\beta q^2}\right|^2\right\}\mathrm{d}\mu\,\mathrm{d}k.$$

One concludes that (28) and (29)–(31) (or equivalently (26) and (27)) follow from the variational principle

$$\delta\left\{Q_1(\Phi_1) + Q_2(\Phi_2) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{\mathrm{nl}}(\Phi_1+\Phi_2)\,\mathrm{d}x\,\mathrm{d}z\right\} = 0. \tag{32}$$

The following lemma describes another aspect of the coupled system (28) and (29)–(31) (which is independent of its variational structure).

Lemma 5.2. *Equation (27) is locally solvable and yields the strictly non-linear functional relationship* $\Phi_2 = \Phi_2(\Phi_1)$.

Lemma 5.2 and the above variational characterisation of (28) and (29)–(31), in which the quadratic part of the variational functional decouples into separate forms for Φ_1 and Φ_2, are the elements of a standard theory in analysis (see Mielke[13]), which forms for example the theoretical basis of the variational Lyapunov-Schmidt reduction. This theory asserts that the 'bifurcation equation'

$$\frac{\epsilon^2}{1+\epsilon}\Big(- c_0(\epsilon^4\partial_z^6 + 3\epsilon^3\partial_x^2\partial_z^4 + \epsilon^2\partial_x^4\partial_z^2 + \epsilon\partial_x^6)\Phi_1$$
$$+ (\beta - 1/3)(\epsilon^2\partial_z^4 + 2\epsilon\partial_x^2\partial_z^2 + \partial_x^4)\Phi_1 - (1+\epsilon)\partial_z^2\Phi_1 - \partial_x^2\Phi_1\Big)$$
$$= \left(\int_0^1 \mathcal{N}_1(\Phi_1 + \Phi_2(\Phi_1))\,\mathrm{d}\zeta + \mathcal{N}_2(\Phi_1 + \Phi_2(\Phi_1))\right)$$

obtained by inserting $\Phi_2 = \Phi_2(\Phi_1)$ into equation (28) follows from the variational principle

$$\delta\left\{Q_1(\Phi_1) + Q_2(\Phi_2(\Phi_1)) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{\mathrm{nl}}(\Phi_1 + \Phi_2(\Phi_1))\,\mathrm{d}x\,\mathrm{d}z\right\} = 0$$

obtained by inserting $\Phi_2 = \Phi_2(\Phi_1)$ into (32). This bifurcation equation is *semilinear* since its highest-order derivatives (sixth-order) occur in the linear part, and the highest-order derivatives in the corresponding variational principle (third-order) occur in the quadratic part $Q_1(\Phi_1)$. We have therefore reduced the original quasilinear problem into a locally equivalent semilinear problem to which well-established techniques from the calculus of variations may be applied. A discussion of these techniques in the context of a model equation for water waves is given by Groves[16]; a similar strategy is used for the present problem (which has some additional technical difficulties, in particular the fact that it involves nonlocal operators).

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