

# The pole dynamics of rational solutions of the Burgers equation

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$$u_t + uu_x = \nu u_{xx}.$$

Some properties:

- Dissipative, not Hamiltonian ( $\nu$ ).
- Linearizable, through the Forsyth-Cole-Hopf transformation:

$$u = -2\nu\phi_x/\phi,$$

so that  $\phi_t = \nu\phi_{xx}$ .

## 2 Rational solutions

By definition, a rational solution with  $N$  poles is of the form

$$u(x, t) = -2\nu \sum_{k=1}^N \frac{R_k(t)}{x - x_k(t)},$$

where  $x_k(t)$  denotes the time-dependent position of a pole, and  $R_k(t)$  its residue. This representation is valid as long as no poles collide.

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Substitution in the Burgers equation gives

$$R_k(t) \equiv 1,$$

for all  $k$ .

We also find

$$\dot{x}_k = -2\nu \sum_{n \neq k}^N \frac{1}{x_k - x_n}, \quad k = 1, \dots, N.$$

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Remarks:

- **local direction field:**  $z' = -2\nu/z$ : poles attract horizontally, repel vertically.
- **Literature:** Chudnovski'i & Chudnovski'i (KdV, Burgers, ...), Senouf et al (Infinite number of poles).

### 3 A complete set of constants of the motion

Define

$$J_n(t) = \frac{1}{n} \sum_{k=1}^N x_k^n(t), \quad n = 1, \dots, N.$$

Note that the quantities  $J_n(t)$  are functionally independent.

**Theorem** The quantities  $J_1, J_2, \dots, J_N$  are polynomial in  $t$ . This  $t$  dependence is completely determined by the recursion relationship

$$\frac{dJ_n}{dt} = -\nu(n-2)(2N+1-n)J_{n-2} - \nu \sum_{k=1}^{n-3} k(n-2-k)J_k J_{n-2-k},$$

and  $J_1 = J_{10}$ ,  $J_2 = -\nu N(N-1)t + J_{20}$ .

For  $n = 3$  the sum in the recursion relationship is to be ignored.

Here  $J_{10}, J_{20}, \dots, J_{N0}$  are constants of the motion, determined by the initial position of the poles

$x_1, x_2, \dots, x_N$ , such that  $J_n(0) = J_{n0}$ ,  $n = 1, 2, \dots, N$ .

For example, we have

$$J_1 = J_{10},$$

$$J_2 = J_{20} - \nu N(N - 1)t,$$

$$J_3 = J_{30} - 2\nu(N - 1)J_{10}t,$$

$$J_4 = J_{40} - \nu \left( 2(2N - 3)J_{20} + J_{10}^2 \right) t + \nu^2 N(N - 1)(2N - 3)t^2,$$

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Thus, there is a complete set of functionally independent conserved quantities, thus the system governing the pole dynamics is integrable.

## 4 Similarity solutions

Let

$$x_n(t) = x_0 + T(t)\zeta_n, \quad n = 1, \dots, N,$$

with  $\zeta_k, x_0 \in \mathbb{C}$ . We get

$$TT' = -\frac{2\nu}{\zeta_k} \sum_{n \neq k}^N \frac{1}{\zeta_k - \zeta_n}, \quad k = 1, \dots, N.$$

Since the right-hand side does not depend on  $t$ , and the left-hand side is independent of the index  $k$ , both sides are equal to a constant.

We get the algebraic equations

$$T = \sqrt{2\nu(t - t_0)},$$

$$\zeta_k = -2 \sum_{n \neq k}^N \frac{1}{\zeta_k - \zeta_n}, \quad k = 1, \dots, N.$$

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**Examples.**

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## Examples.

- $u = \frac{-4\nu x}{x^2 + 2\nu(t - t_0)}.$
- $u = \frac{-4\nu x}{x^2 + 2\nu(t - t_0)(3 - \sqrt{6})} + \frac{-4\nu x}{x^2 + 2\nu(t - t_0)(3 + \sqrt{6})}.$

## 5 Stability of the similarity solutions

First we transform to new variables, where the similarity solutions correspond to stationary solutions:

$$x_j = x_0 + \sqrt{2(t - t_0)} z_j(t), \quad j = 1, \dots, N.$$

Then  $z_j(t) = \zeta_j$ ,  $j = 1, \dots, N$  gives the similarity solutions.

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The  $z_j$ 's satisfy

$$\frac{dz_j}{d\tau} = -\nu \sum_{k \neq j}^N \frac{1}{z_j - z_k} - \frac{1}{2} z_j, \quad j = 1, \dots, N.$$

Here  $\tau = \ln(t - t_0)$ .

## Plan of attack:

- Prove asymptotic linear stability of the similarity solutions
- Infer nonlinear stability from Hartman-Grobman.

So, first we linearize around  $z_j = \zeta_j$ ,  $j = 1, \dots, N$ . Let

$$z_j = \zeta_j + \epsilon \xi_j + \mathcal{O}(\epsilon^2), \quad j = 1, \dots, N.$$

Then

$$\begin{aligned} \dot{\xi}_j &= \left( -\frac{1}{2} + \nu \sum_{k \neq j}^N \frac{1}{(\zeta_j - \zeta_k)^2} \right) \xi_j + \\ &\quad - \nu \sum_{k \neq j}^N \frac{\xi_k}{(\zeta_j - \zeta_k)^2}, \quad j = 1, \dots, N. \end{aligned}$$

Next, let

$$\zeta_j = x_0 + i\beta_j, \quad \beta_j \in \mathbb{R}, \quad j = 1, \dots, N,$$

so that

$$\begin{aligned} \dot{\xi}_j &= \left( -\frac{1}{2} - \nu \sum_{i \neq j}^N \frac{1}{(\beta_j - \beta_i)^2} \right) \xi_j + \\ &\quad \nu \sum_{k \neq j}^N \frac{\xi_k}{(\beta_j - \beta_k)^2}, \quad j = 1, \dots, N. \end{aligned}$$

Now we can read off the matrix  $\Lambda$  of the linear system

$$\xi' = \Lambda \xi,$$

where  $\xi = (\xi_1, \dots, \xi_N)^T$ :

$$\Lambda_{jk} = \begin{cases} -\frac{1}{2} - \nu \sum_{i \neq j}^N \frac{1}{(\beta_j - \beta_i)^2} & \text{if } k = j, \\ \nu \frac{1}{(\beta_j - \beta_k)^2} & \text{if } k \neq j, \end{cases}$$

for  $j, k = 1, \dots, N$ .

## The spectrum of $\Lambda$ :

Define

$$r_j = \nu \sum_{k \neq j}^N \frac{1}{(\beta_i - \beta_j)^2} > 0, \quad j = 1, \dots, N.$$

Then, from Gershgorin's theorem, all eigenvalues are contained in

$$\left| \lambda - \left( -\frac{1}{2} - r_j \right) \right| \leq r_j, \quad j = 1, \dots, N.$$

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Thus, all eigenvalues are to the left of or at  $-1/2$ .

Further, since  $\Lambda$  is symmetric, all eigenvalues are real, thus

$$\sigma(\Lambda) \subset \bigcup_{j=1}^N \left[ -\frac{1}{2} - 2r_j, -\frac{1}{2} \right].$$

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This solution corresponds to

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} e^{-\tau/2} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (t - t_0)^{-1/2}$$

- Since  $\Lambda$  has a complete set of eigenvectors (symmetry), all other fundamental solutions decay faster than  $e^{-\tau/2}$ , corresponding to decay in  $t$ .

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So there.