

The (in)stability of Stationary Periodic Solutions of Integrable Equations

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Outline

- General method (Example: KdV)
 1. Stationary periodic solutions of Nonlinear PDEs
 2. Stability of solutions: Nonlinear-Orbital, Linear, Spectral
 3. Spectral Stability using the squared eigenfunction connection
 4. Nonlinear Stability using Krein signatures

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- Non-stationary periodic solutions, conjectures

Part I

General method

1 Stationary periodic solutions of Nonlinear PDEs

Consider a nonlinear PDE

$$u_t = \mathcal{N}(u, u_x, \dots).$$

A stationary solution

$$u(x, t) = U(x - Vt)$$

is determined by the ODE

$$-VU' = \mathcal{N}(U, U', \dots).$$

We are interested in solutions of the ODE that are periodic in x .

Example. The KdV equation is given by

$$u_t + uu_x + u_{xxx} = 0.$$

Its stationary periodic solutions are given by

$$u = u_0 + 12k^2\kappa^2\text{cn}^2(\kappa(x - x_0 - (8\kappa^2k^2 - 4\kappa^2 + u_0)t), k),$$

with parameters u_0 , κ , x_0 and k . Here

$$V = 8\kappa^2k^2 - 4\kappa^2 + u_0.$$

2 Stability of solutions: Nonlinear-Orbital, Linear, Spectral

Introducing

$$y = x - Vt, \tau = t,$$

we have that stationary solutions $u(x, t) = U(y)$ are fixed points of

$$u_\tau = Vu_y + \mathcal{N}(u, u_y, \dots). \quad (*)$$

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$$u_\tau = Vu_y + \mathcal{N}(u, u_y, \dots). \quad (*)$$

A fixed-point solution $u(y, \tau) = U(\tau)$ of (*) is (nonlinearly) stable if

$$\forall \epsilon > 0, \exists \delta > 0 : \|U(\tau) - u(y, 0)\| < \delta \Rightarrow \|U(\tau) - u(y, \tau)\| < \epsilon.$$

To simplify, we can consider infinitesimal perturbations: let

$$u(y, \tau) = U(y) + \varepsilon w(y, \tau) + \mathcal{O}(\varepsilon^2).$$

Then, to first order in ε , $w(y, \tau)$ satisfies

$$w_\tau = \mathcal{L}(y)w.$$

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A fixed-point solution $u(y, \tau) = U(\tau)$ of (*) is **linearly stable** if

$$\forall \epsilon > 0, \exists \delta > 0 : \|w(y, 0)\| < \delta \Rightarrow \|w(y, \tau)\| < \epsilon.$$

Example. For KdV we get

$$u_\tau - Vu_y + uu_y + u_{yyy} = 0.$$

Linearizing gives

$$w_\tau = \mathcal{L}w,$$

with

$$\mathcal{L} = -\partial_y^3 + (V - U)\partial_y - U_y = \partial_y \left(-\partial_y^2 + V - U \right).$$

Since $U(y)$ does not depend on τ , \mathcal{L} is independent of τ , and we may separate variables. Let

$$w(y, \tau) = W(y, \lambda)e^{\lambda\tau}.$$

Then

$$\lambda W = \mathcal{L}W,$$

which is a spectral problem. The **stability spectrum** $\sigma(\mathcal{L})$ of \mathcal{L} is defined as

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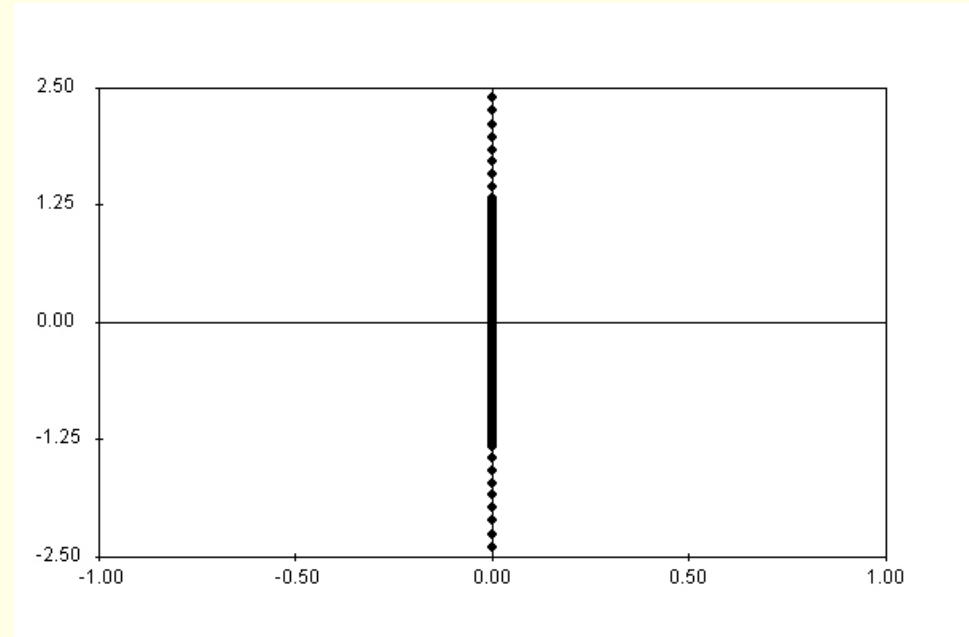
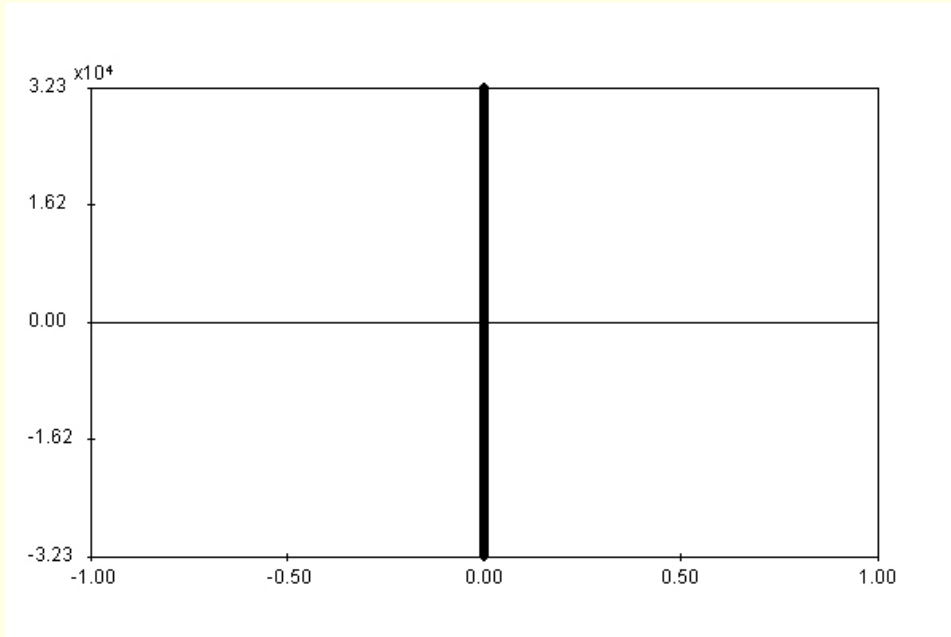
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$$\sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} : \exists \|W(y, \lambda)\| < \infty\}.$$

A fixed-point solution $u(y, \tau) = U(\tau)$ of (*) is **spectrally stable** if

$$\sigma(\mathcal{L}) \cap \text{RHP} = \emptyset.$$

Example. For KdV and its cnoidal wave solution,



(left) The numerically computed spectrum with $k = 0.8$, $\kappa = 1$, $x_0 = 0$ and $u_0 = 0$, using Hill's method with $N = 40$ (81 Fourier modes) and $D = 50$ (49 different Floquet exponents); (right) A blow-up of (a) around the origin, showing a band of higher spectral density.

3 Spectral stability using the squared eigenfunction connection

3.1 The Lax spectrum of the stationary solution

The equation

$$u_\tau = V u_y + \mathcal{N}(u, u_y, \dots)$$

has a Lax pair (of order N), given by

$$\psi_y = Y(\zeta, y)\psi, \quad \psi_\tau = T(\zeta, y)\psi,$$

where we have restricted to the stationary solution.

The **Lax spectrum** σ_L is defined as

$$\sigma_L = \{\zeta \in \mathbb{C} : \exists \|\psi(y, \tau, \zeta)\| < \infty\}.$$

We can separate variables on the second equation:

$$\psi(y, \tau) = e^{\Omega\tau} \Psi(y, \Omega),$$

where Ω is determined by

$$\det(T(y, \zeta) - \Omega I) = 0,$$

which specifies Ω as an N -valued function of ζ .

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Next

$$(T(y, \zeta) - \Omega I) \Psi(y, \Omega) = 0,$$

which determines Ψ up to a multiplicative y -dependent scalar:

$$\Psi(y, \Omega) = \gamma(y, \Omega) \hat{\Psi}(y, \Omega).$$

Substitution in $\psi_y = Y(\zeta, y)\psi$ determines γ as the solution of a first-order linear homogeneous ODE:

$$\gamma = \gamma_0(y) e^{\int S(\zeta, \Omega, y) dy}.$$

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A judicious choice of γ_0 ensures that $\gamma_0(y)\hat{\Psi}(y, \Omega)$ is bounded in y . Thus, to ensure that $\psi(y, \tau)$ is bounded as a function of y we need

$$\operatorname{Re} \langle S(\zeta, \Omega, y) \rangle = 0,$$

with $\langle \cdot \rangle = \frac{1}{T} \int_0^T \cdot dy$. This equation determines the Lax spectrum σ_L explicitly.

Example. In the (y, τ) coordinates, the Lax pair for KdV becomes

$$\psi_y = \begin{pmatrix} 0 & 1 \\ \zeta - U/6 & 0 \end{pmatrix} \psi,$$

$$\psi_\tau = \begin{pmatrix} U_y/6 & -4\zeta + V - U/3 \\ -4\zeta^2 + \zeta V + (U^2 - 3VU + 6\zeta U + 3U_{yy})/18 & -U_y/6 \end{pmatrix} \psi,$$

where we have restricted to the cnoidal wave.

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Let

$$\psi(y, \tau) = e^{\Omega\tau} \begin{pmatrix} \alpha(y) \\ \beta(y) \end{pmatrix},$$

then

$$\begin{pmatrix} U_y/6 - \Omega & -4\zeta + V - U/3 \\ -4\zeta^2 + \zeta V + (U^2 - 3VU + 6\zeta U + 3U_{yy})/18 & -U_y/6 - \Omega \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

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Thus Ω is determined by

$$\Omega^2 = 16(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3),$$

with

$$\zeta_1 = k^2 - 1 \leq \zeta_2 = 2k^2 - 1 \leq \zeta_3 = k^2.$$

The eigenvector $(\alpha, \beta)^T$ is given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma(y) \begin{pmatrix} 4\zeta - V + U/3 \\ U_y/6 - \Omega \end{pmatrix}.$$

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The scalar function $\gamma(y)$ is determined from the first Lax equation:

$$\gamma = \gamma_0 \exp \left(\int \frac{-U_y/6 - \Omega}{4\zeta - V + U/3} dy \right).$$

The condition determining the Lax spectrum σ_L is

$$\operatorname{Re} \left\langle \left(\frac{-U_y/6 - \Omega}{4\zeta - V + U/3} \right) \right\rangle = 0,$$

Thus, the Lax spectrum is the subset of the real ζ -line for which the previous equation holds. We obtain

$$\sigma_L = (-\infty, k^2 - 1] \cup [2k^2 - 1, k^2].$$

Once σ_L is determined, the set of all Ω values follows immediately from

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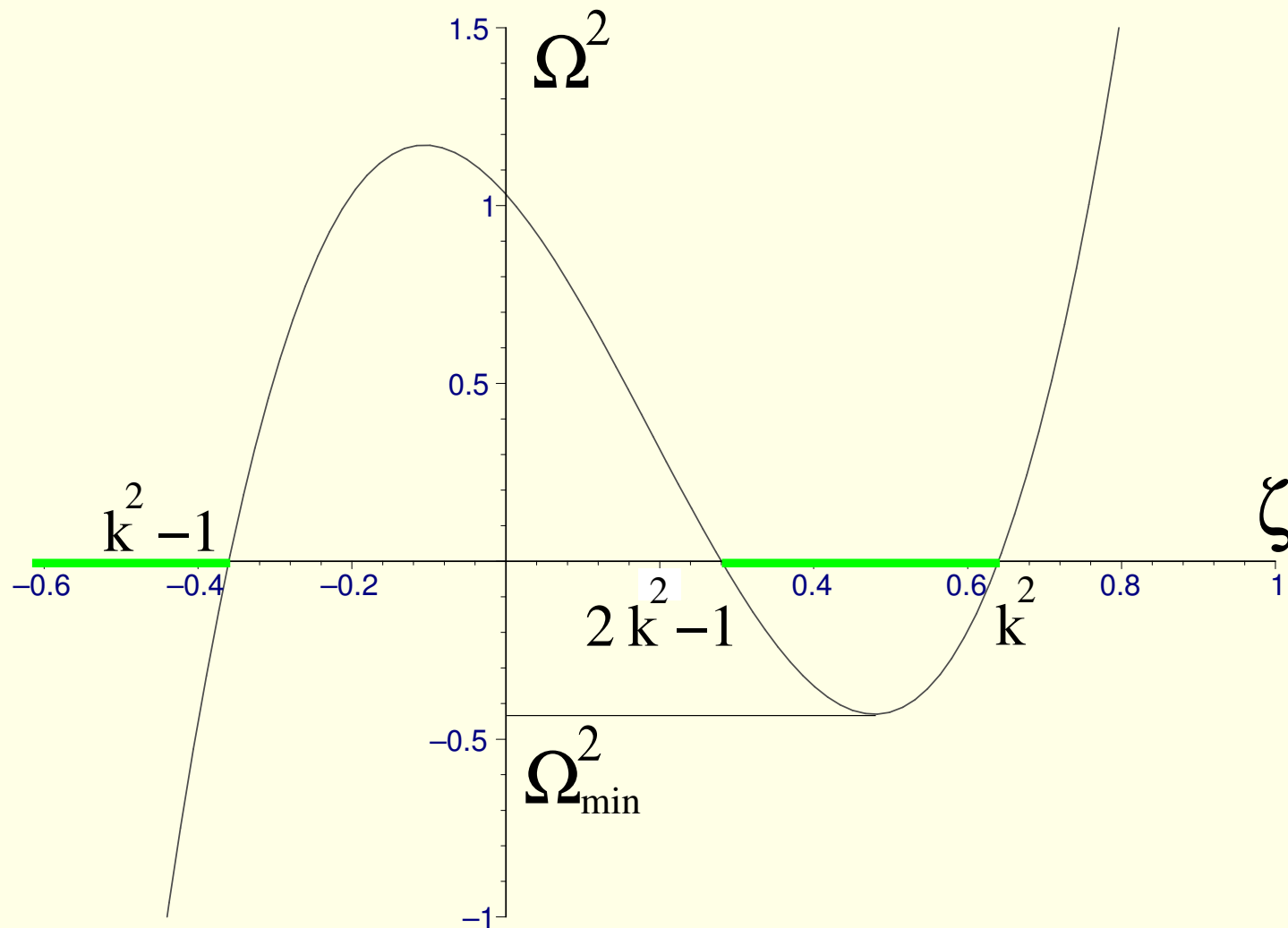
Example. For KdV, the set of all corresponding Ω values is

$$\sigma_\tau = i\mathbb{R} \cup \left[-i\sqrt{|\Omega_{\min}^2|}, i\sqrt{|\Omega_{\min}^2|} \right]^2,$$

where

$$|\Omega_{\min}^2| = \frac{16}{27}(k^2 + 1 + \Delta)(1 - 2k^2 + \Delta)(k^2 - 2 + \Delta),$$

with $\Delta^2 = 1 - k^2 k'^2 \geq 0$.



The Lax spectrum σ_L , indicated in green, with the implied values of Ω .

4 The squared eigenfunction connection and spectral stability

For all integrable equations, there exists a squared eigenfunction connection relating $w(y, \tau)$ and $\psi(y, \tau)$:

$$w = \sum_{i,j} A_{ij} \psi_i \psi_j,$$

for some matrix A . It follows that $\lambda = 2\Omega$.

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The **stability spectrum** is a scaled version of the set of Ω values.

Example. For KdV, this allows us to conclude spectral stability of the cnoidal wave solution.

Linear stability

To infer linear stability from spectral stability, we can use the SCS lemma (Haragus & Kapitula, 2008).

5 Nonlinear stability using Krein signatures

Suppose we have established that the solution $U(y)$ of

$$u_\tau = V u_y + \mathcal{N}(u, u_y, \dots)$$

is linearly stable. Can we conclude it is orbitally stable?

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Suppose we have established that the solution $U(y)$ of

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Orbital stability w.r.t. **subharmonic perturbations**:

$$\begin{aligned} \forall \epsilon > 0, \exists \delta > 0 : \text{if } \|w(y, 0) - U(y)\| < \delta \\ \Rightarrow \inf_{\omega} \|w(y, t) - U(y + \omega)\| < \epsilon. \end{aligned}$$

We approach this problem using the work of Grillakis, Shatah and Strauss (1990) and Maddocks and Sacks (1993).

The spectral stability problem can be written as

$$JLW = \lambda W,$$

where we assume the spectrum of $JL = \mathcal{L}$ is on the imaginary axis.

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How many of these eigenvalues have negative Krein signature?

$$K_1(\lambda) = \langle W, LW \rangle_n = \int_{-nT/2}^{nT/2} W^* LW dy$$

Example. For KdV

$$K_1(\lambda) = \lambda(\zeta)^2 \int_{-nT/2}^{nT/2} (\zeta - k^2 - k^2 \operatorname{sn}^2(y, k) + 1) dy.$$

1. For $\zeta < k^2 - 1$ (simple covering of $i\mathbb{R}$) we see
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1. For $\zeta < k^2 - 1$ (simple covering of $i\mathbb{R}$) we see $K_1(\lambda) > 0$,
2. but for $\zeta \in (2k^2 - 1, k^2)$ (double covering around the origin), we have $K_1(\lambda) < 0$.

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No conclusion!

We may calculate many other Krein signatures, related to the other equations of the nonlinear hierarchy of

$$u_\tau = V u_y + \mathcal{N}(u, u_y, \dots).$$

Denote these equations by

$$\begin{aligned} u_{\tau_k} &= J \frac{\delta}{\delta u} \left(H_k(u, u_y, \dots) + \sum_{j=0}^{k-1} c_j H_j(u, u_y, \dots) \right) \\ &= J \frac{\delta \mathcal{H}_k}{\delta u}, \end{aligned}$$

where H_k is the k -th conserved quantity for our integrable equation.

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- The related stability problem is

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- The related Krein signature is

$$\begin{aligned}
 K_k(\lambda) &= \left\langle W, \left(L_k + \sum_{j=0}^{k-1} c_j L_j \right) W \right\rangle_n \\
 &= \int_{-nT/2}^{nT/2} W^* \left(L_k + \sum_{j=0}^{k-1} c_j L_j \right) W dy.
 \end{aligned}$$

If for a certain k we can choose the constants c_j so that K_k is positive definite, we may conclude that

- $U(y)$ is orbitally stable w.r.t. subharmonic perturbations under the τ_k dynamics, i.e., there exists a Lyapunov functional.
- Since the different flows commute and share conserved quantities, it follows that $U(y)$ is orbitally stable w.r.t. subharmonic perturbations under the $\tau = t$ dynamics.

Example. For the KdV equation

$$K_2(\lambda) = -(4\zeta + c_1 + 8k^2 - 4)K_1(\lambda).$$

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If we choose c_1 such that

$$4(2 - 4k^2) < c_1 < 4(2 - 3k^2),$$

then

$$K_2(\lambda) \geq 0,$$

with $k_2(\lambda) = 0$ only when $\lambda = 0$.

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Part II

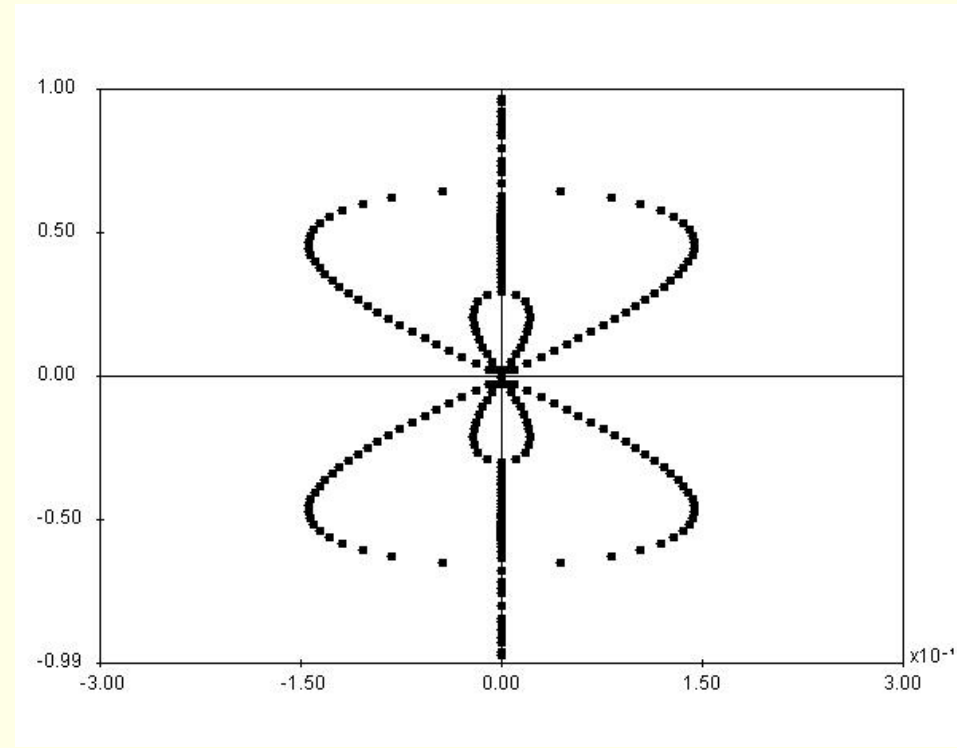
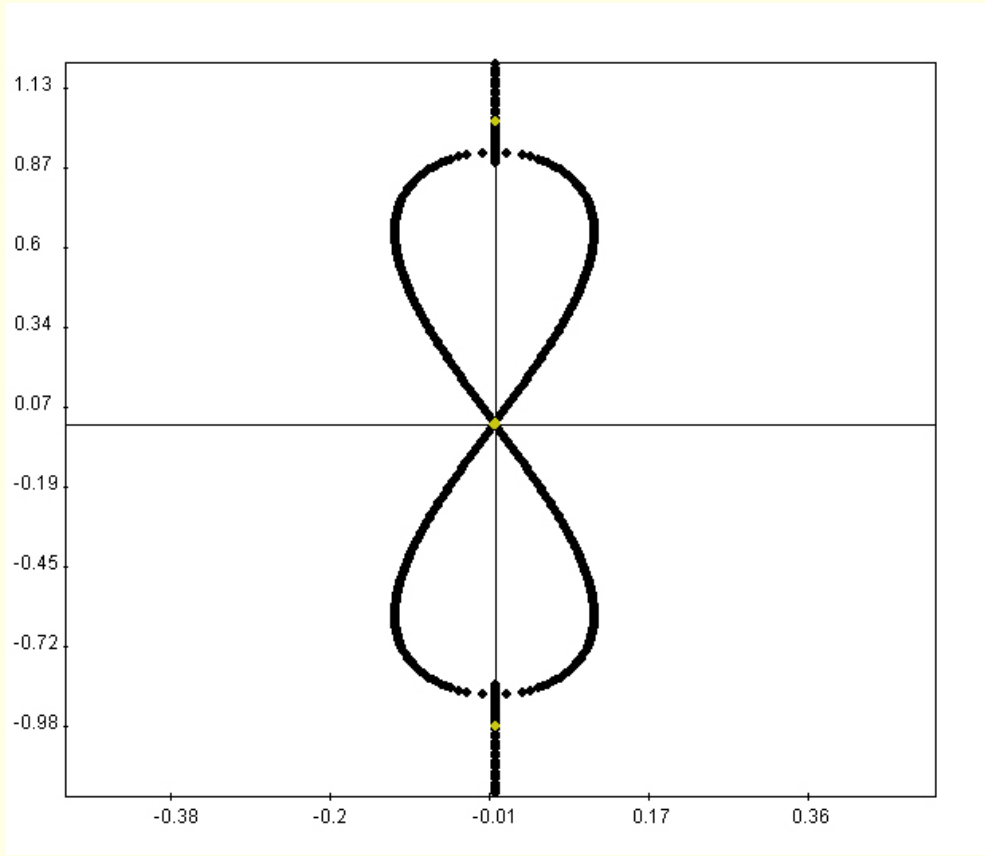
Second Example: NLS

Consider the Nonlinear Schrödinger equation

$$iu_t = -\frac{1}{2}u_{xx} + \alpha u|u|^2.$$

- $\alpha = -1$: focusing NLS
- $\alpha = 1$: defocusing NLS

6 Focusing NLS



The numerically computed spectrum with $k = 0.8$, using Hill's method; The spectrum for the trivial-phase cn solution (left). The spectrum for a nontrivial-phase solution (right). Conclusion: **unstable**.

7 Defocusing NLS

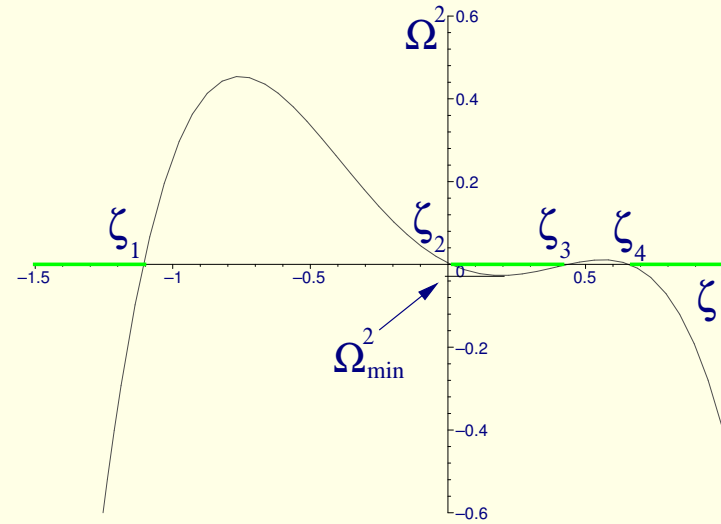
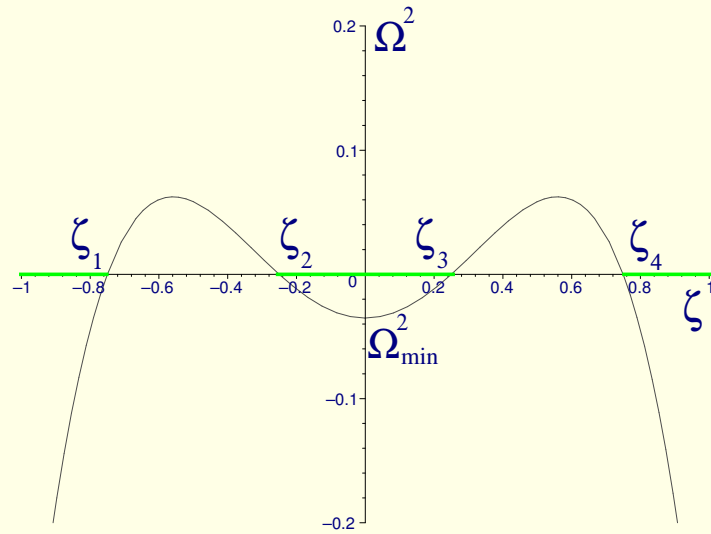
Stationary solutions

$$u = e^{-i\omega t + i\theta(x)} R(x),$$

with

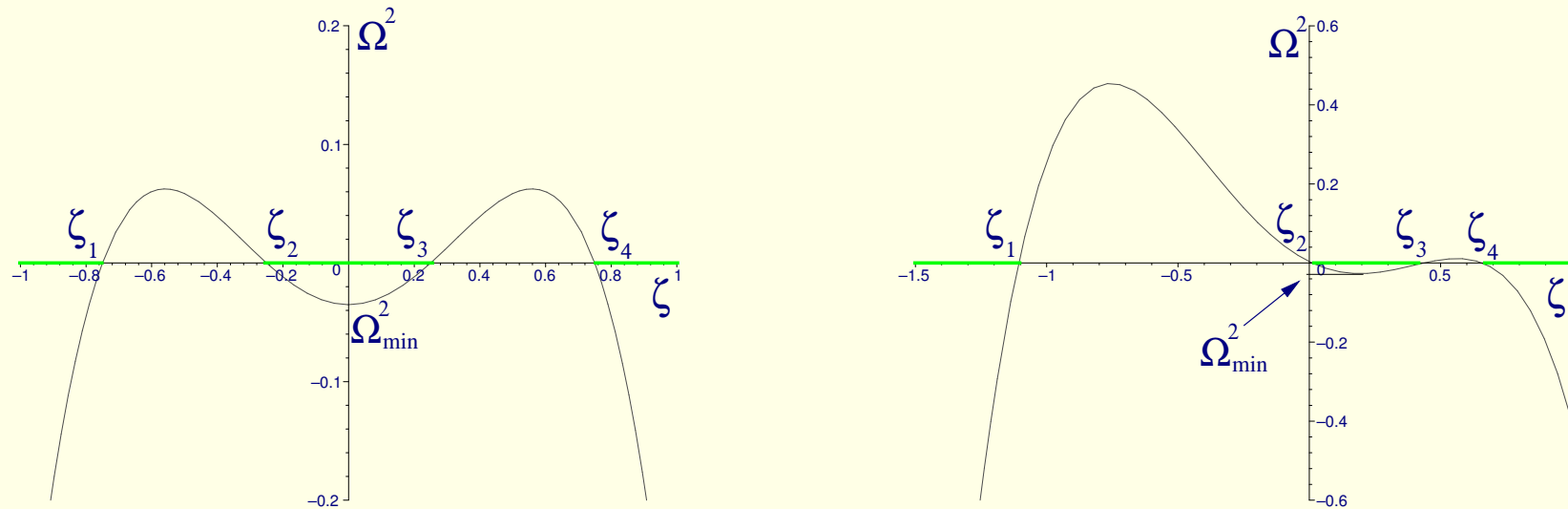
- $R^2 = k^2 \operatorname{sn}^2(x, k) + b,$
- $\theta = \pm \sqrt{b(b+1)(b+k^2)} \int_0^x 1/R^2(x) dx,$
- $\omega = \frac{1}{2}(1+k^2) + 3b/2.$

The Lax spectrum and the corresponding Ω values



Ω^2 as a function of real ζ , for $k = 0.5$. The union of the green line segments is the Lax spectrum σ_L . The figure on the left shows the symmetric trivial-phase case with $b = 0$. The figure on the right illustrates a nontrivial-phase case, with $b = 0.2$.

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Ω^2 as a function of real ζ , for $k = 0.5$. The union of the green line segments is the Lax spectrum σ_L . The figure on the left shows the symmetric trivial-phase case with $b = 0$. The figure on the right illustrates a nontrivial-phase case, with $b = 0.2$.

Spectral stability!

Orbital stability

$$K_1(\lambda) = -2\lambda(\zeta)^2 \int_{-nT/2}^{nT/2} \left(2\zeta^2 + p^2 + q^2 - \omega \right) dx,$$

where

$$R(x)e^{i\theta(x)} = e^{i\eta x} (p(x) + iq(x)).$$

Since there are two sign changes for K_1 at $\zeta = \pm\zeta_0$ it does not suffice to consider K_2 .

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Since there are two sign changes for K_1 at $\zeta = \pm\zeta_0$ it does not suffice to consider K_2 .

Considering K_3 :

$$K_3 = c_2 c_3 (\zeta - \zeta_{32} + c_2) (\zeta - \zeta_{31} + c_1) K_1$$

Choosing

$$c_1 = \zeta_{31} + \zeta_0, \quad c_2 = c_3 = \zeta_{32} - \zeta_0,$$

we have

$$K_3(\lambda) \geq 0,$$

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Part III

Non-stationary solutions, conjectures

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- Also done: MKdV, focusing and defocusing.
- Using higher-order flows, we can prove the stability of higher-genus solutions
- Conjecture: if the Lax operator of the integrable equation is self-adjoint, the stationary solutions are spectrally and orbitally stable.