AMath 401
Homework #6 Due November 29, 2005

While collaboration is allowed (even encouraged), every student must write up their own homework alone. Show all work.

Note the change of due date; 2 pages (6 problems).

Integration and Power Series Representations of Complex Functions

1. Evaluate the following integrals using explicit parametrization, anti-derivatives, the Cauchy-Goursat theorem or Cauchy's integral formula as appropriate.

   (a) $\oint_C \cosh z \, dz$, C is the positive closed contour $|z| = 3$.

   (b) $\int_C \bar{z}^2 \, dz$, C is the circular arc $|z| = 1$, $0 \leq \arg z \leq \pi/2$.

   (c) $\oint_C \frac{e^z}{z^2 - 7z + 6} \, dz$, C is the positive closed contour $|z| = \pi$.

   (d) $\int_C z^{1/2} \, dz$, C is the spiral $z(t) = e^{-t+i\frac{3\pi}{2}t}$, $0 \leq t \leq 1$.

2. Bounds on integrals.

   (a) Use bounding arguments to show that

   $$\lim_{R\to\infty} \int_{C_R} \frac{z + \operatorname{Log} z}{z^3 + 1} dz = 0$$

   where C_R is the arc of the circle $|z| = R$ with $-\pi/2 \leq \arg z \leq \pi/2$.
   Note: $\operatorname{Log} z$ is the principal branch of $\log z$, *i.e.*, $\operatorname{Log} z = \ln r + i\theta$, $-\pi < \theta \leq \pi$.

   (b) Show that

   $$\left| \int_C \frac{z^2 - 1}{z^2 + 1} \, dz \right| \leq \pi R \frac{R^2 + 1}{R^2 - 1}$$

   where C is a semicircle of radius $R > 1$ centered at the origin.

3. (a) The function $u(x, y)$ is harmonic (*i.e.*, $\nabla^2 u = 0$) inside a simple closed contour C. Show that u cannot have a maximum or minimum *inside* C. (Hint: consider e^f and e^{-f} for $f = u + iv$ and use the maximum modulus principal.)

   (b) $f(z)$ is analytic inside and on a simple closed contour C which contains $z = 0$. On C, $|f(z)| > M$. Furthermore, $|f(0)| < M$. Show that $f(z)$ has *at least* one zero inside C. (Hint: consider $1/f$.)
   Note: we can use this to deduce that every polynomial has a root.

4. Show that if $f(z)$ is of the form

$$f(z) = \frac{\alpha_k}{z^k} + \frac{\alpha_{k-1}}{z^{k-1}} + \cdots \frac{\alpha_1}{z} + g(z)$$

where $g(z)$ is analytic inside and on the closed contour $C : |z| = 1$, then

$$\oint_C f(z)\, dz = 2\pi i \alpha_1.$$

Hint: Use Cauchy-Goursat and parameterization as appropriate. Note: α_1 is known as the *residue* of $f(z)$ at $z = 0$.

5. Find the circle of convergence of the following series:

$$\text{(a)} \sum_{k=1}^{\infty} k^k z^k, \quad \text{(b)} \sum_{k=1}^{\infty} \frac{k!}{k^k} z^k, \quad \text{(c)} \sum_{k=0}^{\infty} (k+1)^2 (z+5i)^{2k}, \quad \text{(d)} \sum_{k=0}^{\infty} \left(k + 2^k\right) z^k$$

6. Let $f(z) = \sum_{k=1}^{\infty} k^3 (z/4)^k$. Evaluate the following contour integrals:

   (a) $\displaystyle\oint_C \cos(iz) f(z)\, dz,$ $\qquad C$ is the postively oriented circle $|z - 1| = 1$.

   (b) $\displaystyle\oint_C \frac{f(z)}{z^3} dz,$ $\qquad C$ is the postively oriented circle $|z| = \pi$.

   Hint: determine the circle of convergence of the series first.
   Note: since a power series is guarenteed to be *uniformly* convergent wherever it is convergent, it can be integrated term-by-term.