Advanced Calculus 2003

Spring 2003

I. Infinite Series

(1) [10pt] Find the sum of the infinite series, with justification

$$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + ...$$

(2) [10pt] Suppose that $x_1, x_2, x_3, \ldots, x_k$ are positive numbers satisfying the following conditions:

i) $x_1 \geq x_2 \geq x_3 \geq \ldots$;

ii) the series $\sum_{\ell=1}^{\infty} x_{m^\ell}$ diverges, where $m > 1$ is a given integer, x_{m^ℓ} [reads x with subscript m^ℓ] is the m^ℓth term of the original sequence.

Show that the series

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \ldots$$

diverges.

II. Multivariate and Vector Calculus

(3) [10pt] The multivariate function $F(x, y, z, t)$ satisfies the following condition

$$\frac{\partial F}{\partial t} + \mathbf{b} \cdot \nabla F + F \nabla \cdot \mathbf{b} - \nabla^2 F = 0,$$

where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\mathbf{b} = (x, y, z)$. Let

$$G(t) \triangleq \int_{R^3} F(x, y, z, t) dx dy dz$$

be a differentiable function of $t \in [0, \infty)$. What is the function $G(t)$?

(4) [10pt] Suppose $\mathbf{r}(t) = (e^{\alpha t} \cos \alpha t)\mathbf{i} + (e^{\alpha t} \sin \alpha t)\mathbf{j}$. Show that the angle between $\mathbf{r}$ and its acceleration $\ddot{\mathbf{r}}$ never changes. What is the value of this angle?

(5) [10pt] Determine whether or not the given field is conservative.

$$\mathbf{F} = (y/x)\mathbf{i} + (1 + \ln |xy|)\mathbf{j} + z\mathbf{k}$$

If it is, find the potential function f which satisfies $f(1, 1, 0) = 1$.

(6) [10pt] Evaluate the integral

$$\int_C \left(4z + e^x\right) dx + \left(x^3 - \tan^{-1} y\right) dy + \left(y + z^3\right) dz$$

where C is defined as $\mathbf{r}(t) = (\cos t, \sin t, 1)$, $0 \leq t \leq 2\pi$.