

Preliminary Exam
Advanced Calculus
Spring 2004
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1. (20 points) For the following, choose **True** or **False**. You do not need to provide further explanations.

a) Define a surface Γ as the set of points in \mathbb{R}^3 satisfying $F(x, y, z) = 0$. Then ∇F evaluated at a point P_0 of Γ is tangent to Γ at P_0 (assuming that $\nabla F \neq (0, 0, 0)$ at P_0).

b) Let $x_1, x_2, \dots, x_n \in \mathbb{R}$, all positive. Then

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

c) For all 3-vectors \mathbf{a}, \mathbf{b} we have $\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 = \|\mathbf{a} \times \mathbf{b}\|^2 + (\mathbf{a} \cdot \mathbf{b})^2$

d) Let \mathbf{f} and \mathbf{g} be two vector functions. Then $\nabla \cdot (\mathbf{f} \times \mathbf{g}) = (\nabla \cdot \mathbf{f}) \times \mathbf{g} + \mathbf{f} \times (\nabla \cdot \mathbf{g})$.

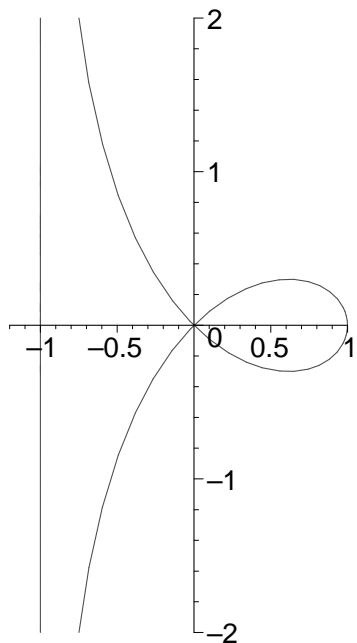
2. (20 points) For what values of x does the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n x^{n-1}}$$

converge? Be sure to consider all possible values of x .

3. (20 points) A customer of your engineering firm want to release a new and improved, FDA-approved headache relief pill. Because the container of the pill should not react with the active ingredients, noble metals have to be used, making this a costly proposition. The customer wants to use a cylindrical container with hemispherical caps, all of constant thickness. This container is supposed to contain a fixed volume of 1cm^3 . How should you design this container to minimize the amount of material used?

4. (20 points)



On the left, you find the graph of the strophoid. This is a curve in \mathbb{R}^2 determined by

$$y^2 = x^2 \frac{\alpha - x}{\alpha + x},$$

for positive α ($\alpha = 1$ for the graph shown on the left). Here x corresponds to the horizontal axis and y corresponds to the vertical axis. The strophoid has a vertical asymptote at $x = -\alpha$.

(a) Using polar coordinates, show that the strophoid can be written as

$$r = \alpha \cos 2\theta \sec \theta.$$

(b) Calculate the area enclosed by the lobe of the strophoid.

5. (20 points)

- Consider

$$I_2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$$

in cartesian coordinates and in polar coordinates. Equating these two, find the value of $J_2 = \int_0^\infty e^{-x^2} dx$ to be $\sqrt{\pi}/2$.

- Consider

$$I_3 = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(x^2+y^2+z^2)} dx dy dz.$$

using both cartesian coordinates and spherical coordinates. Equating these two, find the value of $J_3 = \int_0^\infty e^{-x^2} x^2 dx$ (you may need the result from part a).

(Note: if you prefer to calculate J_2 and/or J_3 another way, that's fine too.)

6. (20 points) For what values of r and θ does the series

$$1 + r \cos \theta + r^2 \cos 2\theta + r^3 \cos 3\theta + \dots$$

converge absolutely? Find the sum of the series in this case.