Preliminary Exam

Advanced Calculus

Spring 2006

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1. (**20 points, 5 points each**)

   a) State the mean-value theorem for differentiable functions.

   b) Give a formula for the derivative of $F(x) = \int_{a(x)}^{b(x)} f(x,t)dt$, where $a(x)$, $b(x)$ and $f(x,t)$ are all differentiable.

   c) Use mathematical induction to prove $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

   d) What is the Taylor series of $x^{1/3}$ around $x = 1$?

2. (**20 points**) Consider the function

$$f(x,y) = e^y(x^2+y^2) - \sqrt{1+y^2} + xy,$$

   where x and y are real variables.

   a) Check that $x = 1$, $y = 0$ solves the equation $f(x,y) = 0$.

   b) Can the equation $f(x,y) = 0$ be solved for y as a function of x near $x = 1$, $y = 0$? Why or why not?

   c) If possible, calculate the first three terms (*i.e.*, a constant, linear and quadratic term) of the Taylor expansion of $y(x)$ near $x = 1$.

3. (**20 points**) Let

$$\boldsymbol{F} = \frac{y\boldsymbol{i} - x\boldsymbol{j}}{x^2+y^2},$$

   where $\boldsymbol{i}$ and $\boldsymbol{j}$ denote the unit vectors in the $x-$ and $y-$ directions.

   a) Calculate $\boldsymbol{\nabla} \times \boldsymbol{F}$

   b) Evaluate $\oint_C \boldsymbol{F} \cdot \boldsymbol{dr}$, where C is *any* closed non-selfintersecting curve in $\mathbb{R}^2$.

4. (**20 points**)

   a) Discuss the convergence of the series $\sum_{n=1}^{\infty} 2^{(-1)^n - n}$.

   b) Discuss $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4+y^2}$.

5. (**20 points**) Consider the curve C, determined by the set of points $(x,y) \in \mathbb{R}^2$ satisfying

$$x = R(t - \sin t), \quad y = R(1 - \cos t),$$

   where $R > 0$ is constant, and $t \in \mathbb{R}$ is a parameter. Examine this curve (*i.e.*, determine whatever properties you deem are relevant; this is purposely open ended) and produce an accurate plot.