Preliminary Exam

Advanced Calculus

Spring 2007

Bernard Deconinck

1. (**10 points**) Find a formula for the shortest distance from a point (x_0, y_0, z_0) to a plane given by $Ax + By + Cz = D$.

2. (**10 points**) Let
$$\boldsymbol{F} = \frac{y\boldsymbol{i} - x\boldsymbol{j}}{x^2 + y^2},$$
where $\boldsymbol{i}$ and $\boldsymbol{j}$ denote the unit vectors in the $x-$ and $y-$ directions.

   a) Calculate $\boldsymbol{\nabla} \times \boldsymbol{F}$

   b) Evaluate $\oint_C \boldsymbol{F} \cdot \boldsymbol{dr}$, where C is *any* closed non-self-intersecting curve in $\mathbb{R}^2$.

3. (**20 points**) In a circle, draw two cords intersecting at right angles. This defines 4 line segments from the point of intersection to the edge of the circle. Denote the lengths of these 4 segments by a, b, c and d. For which configurations is $a^2 + b^2 + c^2 + d^2$ maximal?

4. (**20 points**)

   Consider the series
$$\frac{1}{1^\alpha} + \frac{1}{2^\beta} + \frac{1}{3^\alpha} + \frac{1}{4^\beta} + \ldots,$$
   where $1 < \alpha < \beta$. Let u_n denote the nth term of this series.

   - What is $\lim_{n\to\infty} \left| \frac{u_{2n+1}}{u_{2n}} \right|$?
   - What is $\lim_{n\to\infty} \left| \frac{u_{2n}}{u_{2n-1}} \right|$?
   - Does the series converge? Discuss.

5. (**20 points**) Consider the function $f(x, y)$ on $\mathbb{R}^2$, given by
$$f(x,y) = \begin{cases} 1 & x > 0,\ y > 0,\ 0 \le x - y \le 1, \\ -1 & x > 0,\ y > 0,\ 0 < y - x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   a) Calculate $\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y) dy \right) dx$

   b) Calculate $\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y) dx \right) dy$

   c) Comment on your results.

6. (**20 points**) Consider the curve C, determined by the set of points $(x, y) \in \mathbb{R}^2$ satisfying

$$(x^2 + y^2)^2 = a^2(x^2 - y^2),$$

where $a > 0$ is constant. Examine this curve (*i.e.*, determine whatever properties you deem are relevant; this is purposely open ended) and produce an accurate plot. This curve is known as the Lemniscate of Bernouilli.