Advanced Calculus 2003

I. Infinite Series

(1) [15pt] Find the sum of the infinite series ($r > 0$) with justification

$$\sum_{n=0}^{\infty} \frac{n}{r^n}.$$

(2) [20pt] Fibonacci series

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$$

satisfies the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \qquad (n \geq 2).$$

Find the explicit, general expression for F_n.

II. Multivariate and Vector Calculus

(3) [10pt] Let $x = x(y, z)$, $y = y(x, z)$, and $z = z(x, y)$ be the explicit functions defined by equation $F(x, y, z) = 0$. Compute the product

$$\left(\frac{\partial x}{\partial y}\right)_z \cdot \left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y.$$

(4) [10pt] Let $\vec{r} = -a \sin\theta \vec{i} + a \cos\theta \vec{j} + b\vec{k}$. Find

$$\frac{1}{2} \int_0^{2\pi} \left(\vec{r} \times \frac{d\vec{r}}{d\theta}\right) d\theta.$$

(5) [10pt] Show that for any simple closed loop Γ:

$$\oint_\Gamma 2xzdx + 2yz^2dy + (x^2 + 2y^2z - 1)dz = 0.$$

(6) [15pt] Compute the line integral

$$\oint_\Gamma xy^2dy - x^2ydx$$

where Γ is the circle $x^2 + y^2 = R^2$.

(7) [20pt] Let the volume of the n-dimensional sphere with radius one

$$x_1^2 + x_2^2 + ... + x_n^2 \leq 1$$

be V_n. Show that $V_n = 2\pi V_{n-2}/n$. What is the volume for a n-dimensional sphere with radius R?