Preliminary Exam

Advanced Calculus

Winter 2005

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Answer 5 of the following 6 questions.
Clearly indicate which ones you want graded.

1. (**20 points**) For the following statements, choose **True** or **False**. You do not need to provide further explanations.

   a) Let $u(x,y)$ be infinitely differentiable on $\{(x,y) \in D\} \subset \mathbb{R}^2$. Such a function is called harmonic on D if $\Delta u = 0$. **Statement:** if u is harmonic on D, then so are all of its partial derivatives.

   b) Consider the graph C_n of the equation $x^{2n} + y^{2n} = 1$ in $\mathbb{R}^2$, with $n = 1, 2, 3, \ldots$. This graph is a closed curve in $\mathbb{R}^2$, enclosing the origin $(0,0)$. **Statement:** the area enclosed by C_n decreases as $n \to \infty$.

   c) **Statement:** $1 + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{12^2} + \frac{1}{16^2} + \cdots = \frac{\pi^2}{16}$.

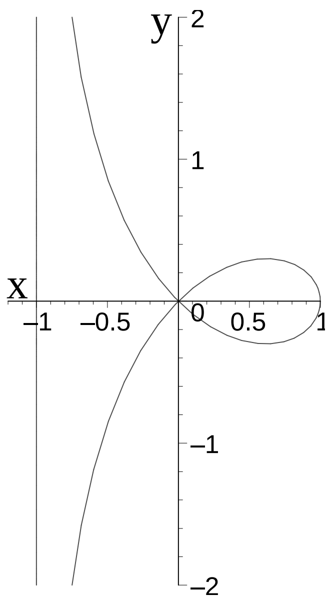
   d) Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be two vector functions. **Statement:** $\nabla \cdot (\boldsymbol{f} \times \boldsymbol{g}) = \boldsymbol{g} \cdot (\nabla \times \boldsymbol{f}) - \boldsymbol{f} \cdot (\nabla \times \boldsymbol{g})$.

2. (**20 points**) Suppose $y(x)$ satisfies the integral equation

$$y(x) = x^2 - 4 + \int_0^{2\pi} t \sin(y(t) - x)dt,$$

for $x \in [0, 2\pi]$. Find an upper bound on $y(x)$ in this interval.

3. (**20 points**)



On the left, you find the graph of the strophoid. This is a curve in $\mathbb{R}^2$ determined by

$$y^2 = x^2 \frac{\alpha - x}{\alpha + x},$$

with α positive. ($\alpha = 1$ for the graph shown on the left). Here x corresponds to the horizontal axis and y corresponds to the vertical axis. The strophoid has a vertical asymptote at $x = -\alpha$. Consider the surface obtained by revolving the strophoid around the x-axis.

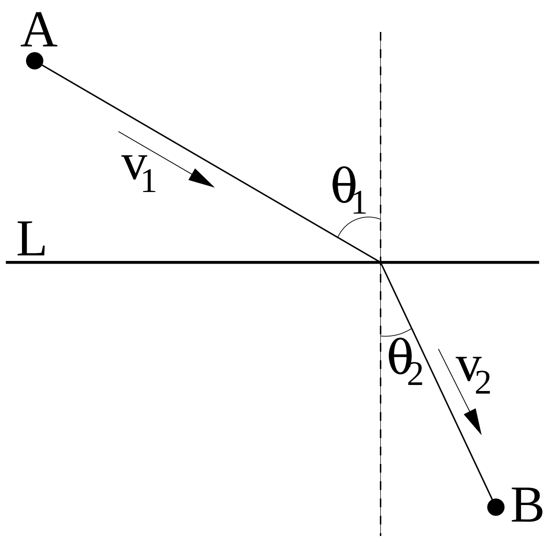
(a) Calculate the volume enclosed by this surface to the right of the y-axis. (*i.e.*, the volume of the droplet)

(b) Calculate the unbounded volume between this surface and the plane $x = -\alpha$.

4. (**20 points**) Give the definition of uniform convergence of a sequence of functions $f_n(x)$ $(n = 1, 2, 3, \ldots)$ to a limit function $f(x)$ on an interval $a \le x \le b$. Are the following sequences of functions $f_n(x)$ uniformly convergent to the given function $f(x)$ on the given interval? Prove your answer.

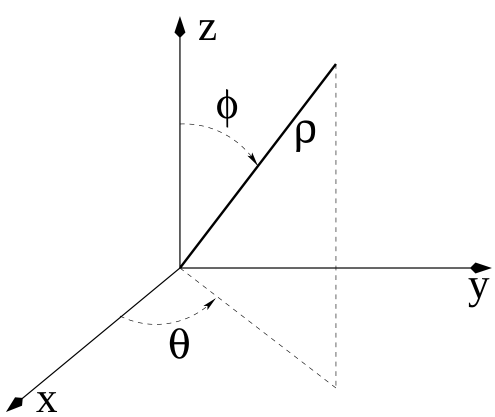
- $f_n(x) = \dfrac{nx}{nx+1}$, $f(x) = 1$, $0 \le x \le 1$.
- $f_n(x) = \displaystyle\sum_{k=1}^{n} \frac{\sin kx}{k^2}$, $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$, $0 \le x \le 2\pi$.

5. (**20 points**)



A particle is to travel in minimal time from point A to point B along a broken straight-line path, as indicated at the left: above the line L, the particle travels with constant velocity v_1 along one straight line, to continue along a (possibly different) straight line below L, also with constant velocity v_2. Show that the motion of the particle is such that $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$. This is Snell's law from linear optics.

6. (**20 points**)



Using spherical coordinates (see left), what kind of figure is defined by $\rho = 2a\cos\phi$ $(a > 0,\ 0 \le \phi \le \pi/2)$? This figure is divided in an upper and a lower part by the plane $z = a$. The attraction of a volume V with density μ on a particle of unit mass at the origin is defined as

$$\vec{F} = \iiint_V \frac{\mu}{r^3}\vec{P}\,dV,$$

where r is the length of $\vec{P}$, and $\vec{P}$ varies over the entire volume. Assuming the density to be constant, (a)what is the attraction of the upper part on the origin? (b) What is the attraction of the lower part on the origin?