

Preliminary Exam
Advanced Calculus
Winter 2006
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1. (20 points, 5 points each)

- a) Give the (ϵ, δ) -definition of $\lim_{x \rightarrow x_0} y(x) = y_0$.
- b) What is the normal vector to the surface defined by $F(x, y, z) = 0$ at the point (a, b, c) , which is on the surface?
- c) State Green's Theorem.
- d) What is the Taylor series of $\sinh x$ around $x = 0$?

2. (20 points) Consider the system of differential equations for a function $\phi(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\begin{cases} \phi_x = A(x, y)\phi, \\ \phi_y = B(x, y)\phi, \end{cases}$$

where the x and y indices denote partial differentiation.

- a) What, if any, are the conditions on A and B so that $z = \phi(x, y)$ describes a smooth surface in \mathbb{R}^3 ? If you found any conditions, are they necessary and/or sufficient?
- b) Let N be a positive integer, greater than 1. Now suppose that $\phi(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}^N$, with $A, B \in \mathbb{R}^{N \times N}$ (i.e., ϕ is an N -dimensional vector function, and A, B are $N \times N$ matrices). What, if any, are the conditions on A, B such that any component of ϕ describes a smooth surface? If you found any conditions, are they necessary and/or sufficient?

3. (20 points) Let

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2},$$

where \mathbf{i} and \mathbf{j} denote the unit vectors in the x - and y - directions.

- a) Calculate $\nabla \times \mathbf{F}$
- b) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is *any* closed non-selfintersecting curve in \mathbb{R}^2 .

4. (20 points) Determine all real x values for which the series

$$S(x) = 2 \sin \frac{x}{3} + 4 \sin \frac{x}{9} + 8 \sin \frac{x}{27} + \dots = \sum_{n=1}^{\infty} 2^n \sin \frac{x}{3^n}$$

converges absolutely. Prove your claim.

5. (20 points) Consider the function $f(x, y)$ on \mathbb{R}^2 , given by

$$f(x, y) = \begin{cases} 1 & x > 0, y > 0, 0 \leq x - y \leq 1, \\ -1 & x > 0, y > 0, 0 < y - x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- a) Calculate $\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x, y) dy \right) dx$
- b) Calculate $\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x, y) dx \right) dy$
- c) Comment on your results.