Preliminary Exam

Advanced Calculus

Winter 2007

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Answer 5 out of 6 questions.
Clearly indicate which one you don't want graded.
If nothing is indicated, question 6 will not be graded.

1. (**20 points, 5 points each**)

   a) True or false:

   $$\lim_{n\to\infty}\left(A_1^n + A_2^n + \ldots + A_k^n\right)^{1/n} = A_1 + A_2 + \ldots + A_k,$$

   where $0 < A_1 < A_2 < \ldots < A_k$, and k is a positive integer.

   b) True or false: let $x \in \mathbb{R}^n$. Let $F(x)$ be a homogeneous function of degree p, *i.e.,* F(x) satisfies $F(\lambda x) = \lambda^p F(x)$, for $\lambda \in \mathbb{R}^+$. Where both sides are defined, we have $x \cdot \nabla F(x) = pF(x)$.

   c) Let $x \in \mathbb{R}^n$. The functions $F_1(x), \ldots, F_k(x)$ are said to be functionally dependent if there exists a function G such that $G(F_1(x), \ldots, F_k(x)) \equiv 0$. True or false: the functions $F_1(x), \ldots, F_k(x)$ are functionally dependent if their gradients $\nabla F_1(x), \ldots, \nabla F_k(x)$ are mutually orthogonal.

   d) Write down the Taylor series of a function $f(x, y)$, analytic in both of its variables, around $(x, y) = (x_0, y_0)$, up to and including second-order terms.

2. (**20 points**) Which is greater, 3^π or π^3? Prove your result.

3. (**20 points**) Let the region D be defined by

   $$\begin{cases} 1/x \le y \le 4x & \text{for} \quad 1/2 \le x \le 1, \\ x \le y \le 4/x & \text{for} \quad 1 \le x \le 2. \end{cases}$$

   - Draw D.
   - State the rule for changing variables in a double integral.
   - Using the transformation $u = y/x$, $v = xy$, calculate

   $$I = \iint_D \frac{4xy^3}{x^2 + y^2}\,dxdy.$$

4. (**20 points**) The island of Prelimac has a coastline given by $2x^2+2xy+5y^2 = 9$. The density of a species of rare, tropical penguins (*Calculinus advanticus*) on the island can be represented by $\rho(x, y) = e^{x^2+2y^2}$. You are a well-educated seal (*Gradius primannius*) and you want to hunt for penguins on the coastline of the island. Find the location(s) on the coast of Prelimac where you should hunt for best results, and the location(s) that would be the least productive.

5. (**20 points**)

- Give the definition of convergence of a series $F(x) = \sum_{n=1}^{\infty} f_n(x)$.
- Give the definition of *uniform* convergence of a series $F(x) = \sum_{n=1}^{\infty} f_n(x)$.
- Why do we care about uniform convergence? What does it do for us?
- Consider the series

$$F(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(1-x^n)(1-x^{n+1})}.$$

Using the partial fractions type identity

$$\frac{1}{(1-x^n)(1-x^{n+1})} = \frac{1}{(1-x)(1-x^n)} - \frac{x}{(1-x)(1-x^{n+1})},$$

calculate $F(x)$ for $|x| < 1$ and for $|x| > 1$. Discuss your findings.

6. (**20 points**) For $y \geq 0$, consider

$$I(y) = \int_0^{\infty} \frac{\sin x}{x} e^{-xy}\, dx.$$

- Calculate $I'(y)$, for $y > 0$. Why does this result not hold for $y = 0$?
- Having found $I'(y) = -1/(1+y^2)$ for $y > 0$, find $I(y)$ for $y > 0$.
- Use a continuity argument to extend your result to $y = 0$, allowing you to determine what

$$\int_0^{\infty} \frac{\sin x}{x}\, dx$$

is.