Advanced Calculus Preliminary Exam

Wednesday, January 9, 2008

Closed book, closed notes, 2 hours. Show your work.

1. (+15)

   (a) (+10) Show that

   $$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{n}{n^2+j^2} = \lim_{n\to\infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+n^2} \right] = \frac{\pi}{4}.$$

   Hint: View the sum as a Riemann sum approximation to some integral.

   (b) (+5) Is the following equality true? Justify your answer.

   $$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{n}{n^2+j^2} = \lim_{N\to\infty} \left[\lim_{n\to\infty} \sum_{j=1}^{N} \frac{n}{n^2+j^2} \right].$$

2. (+10) If $z^3 - xz - y = 0$, show

   $$\frac{\partial^2 z}{\partial x \partial y} = -\frac{3z^2+x}{(3z^2-x)^3}.$$

3. (+20)

   (a) (+5) State Green's Theorem in the plane.

   (b) (+5) Use this to show that the area bounded by a simple closed curve C is given by

   $$\text{Area} = \frac{1}{2} \oint_C (xdy - ydx).$$

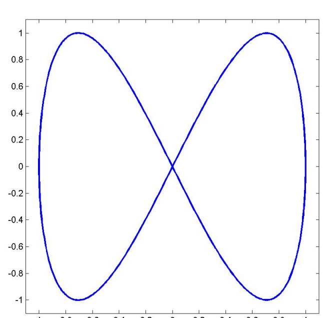
   (c) (+10) The figure-8 curve shown below has the parametric form

   $$x(t) = \cos(t), \quad y(t) = \sin(2t).$$

   Determine the area enclosed by this curve.
   Hint: Integration by parts shows that

   $$\int_{-\pi/2}^{\pi/2} \sin(2t)\sin(t)\,dt = 2\int_{-\pi/2}^{\pi/2} \cos(2t)\cos(t)\,dt.$$

   Another integration by parts allows you to determine the value of this integral.



4. (+20) Let

$$F(x,\alpha) = \frac{x^\alpha - 1}{\ln x}$$

for $0 < x < 1$ and $\alpha > 0$.

(a) (+5) How would you define $F(0,\alpha)$ and $F(1,\alpha)$ so that $F(x,\alpha)$ is continuous in both x and α for $0 \le x \le 1$ and all finite $\alpha > 0$?

(b) (+10) Let

$$\phi(\alpha) = \int_0^1 F(x,\alpha)\,dx.$$

Apply Leibnitz's rule to compute $\phi'(\alpha)$.

(c) (+5) Integrate $\phi'(\alpha)$ to show that $\phi(1) = \ln 2$.

5. (+20) Consider the double integral

$$\int_0^1 \int_0^{1-x} \exp\left(\frac{y}{x+y}\right)\,dy\,dx.$$

(a) (+5) Draw the region D in the xy plane indicated by the limits of integration.

(b) (+5) Consider the change of variables $x + y = u$ and $y = uv$. Draw the region D' in the u-v plane that is the image of D under the transformation. Comment on what you observe.

(c) (+10) Evaluate the double integral by changing variables using the transformation above.

6. (+15) Let $f(x)$ be a smooth, radially symmetric, scalar valued function of $x \in \mathbb{R}^3$, so $f(x) = F(r)$ for some smooth function F, where $r = \|x\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

(a) (+8) Show that

$$\nabla f = \frac{F'(r)}{r} x.$$

(b) (+7) By taking the divergence of the gradient determined in part (a), determine $\nabla^2 f$, the Laplacian of f.