

Preliminary Exam in Linear Algebra
April, 2006

Show all your work.

1. Let \mathcal{P}_k denote the set of all polynomials of degree k or less on the unit interval $0 \leq x \leq 1$. This is a $k + 1$ dimensional linear space since any linear combination of functions in \mathcal{P}_k is again in \mathcal{P}_k . It has a basis consisting (for example) of the monomial functions $1, x, x^2, x^3, \dots, x^k$. These functions span \mathcal{P}_k and are linearly independent, i.e., the only linear combination of these functions that yields the zero function is the trivial linear combination with all coefficients equal to zero.

- (a) Define the inner product of two functions $f(x)$ and $g(x)$ defined on the interval $[0, 1]$ by

$$\langle f, g \rangle = \int_0^1 f(x)g(x) dx.$$

We say f and g are orthogonal if $\langle f, g \rangle = 0$.

Define the 2-norm of f by $\|f\|_2 = \sqrt{\langle f, f \rangle}$.

Find the least squares approximation to the function $f(x) = x^3$ by a linear polynomial $p \in \mathcal{P}_1$ on the interval $[0, 1]$, i.e., find the $p(x)$ that minimizes $\|p - f\|_2$.

- (b) Consider the set of functions $\mathcal{S} = \{p \in \mathcal{P}_2 : \int_0^1 p(x) dx = \int_0^1 p'(x) dx\}$.

Show that this is a linear subspace of \mathcal{P}_2 , determine its dimension and find a basis for \mathcal{S} .

2. Let

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{bmatrix}.$$

- (a) Determine the eigenvalues of A and a basis for each eigenspace.
(b) The matrix A can be diagonalized as $R^{-1}AR = \Lambda$. Determine matrices R and R^{-1} that can be used.
(c) Use the eigenstructure to determine the vector

$$A^{100} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

where A^{100} is the 100th power of the matrix A .

3. Suppose $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times s}$ and both matrices have full column rank. Suppose furthermore that the range of AB is equal to the range of A . What relations (e.g. $<$, \geq , $=$) must hold between the integers m , n , and s ? Carefully explain your reasoning and then give an example that illustrates your conclusions.

4. Let

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 1 & 3 & 5 & 8 \\ 2 & 6 & 10 & 12 \end{bmatrix}.$$

- (a) Determine bases for the range and null space of A .
- (b) Does the system $Ax = b$ with this A and $b = [9 \ 10 \ 16]^T$ have any solutions? If so, determine all solutions.
5. Let A be an $n \times n$ matrix such that $\sum_{j=1}^n |a_{ij}| < 1$ for each i . Prove that $I - A$ is invertible.