Winter 2001

Linear Algebra

1. Consider the linear system

$$\begin{aligned} 6x_2 - x_3 &= 3 \\ 3x_1 + 2x_2 - x_3 &= -7 \\ 6x_1 - 2x_2 &= -14 \end{aligned}$$

$$\begin{pmatrix} 0 & 6 & -1 \\ 3 & 2 & -1 \\ 6 & -2 & 0 \end{pmatrix}$$

(a) Set this up in the form $Ax = b$.

(b) Determine a permutation matrix P so that PA can be factored as LU, the product of a lower and an upper triangular matrix. Compute the LU factorization of the matrix PA.

(c) Use this LU factorization along with forward and back substitution to solve the linear system $PAx = Pb$ and find the solution to the original system above.

2. Let W be the plane in 3-space defined by $W = \{x \in \mathbb{R}^3 : \ x_2 = 2x_3\}$. This is a linear subspace of $\mathbb{R}^3$.

(a) Determine an orthogonal basis for W.

(b) What is the dimension of W?

(c) Find the projection matrix $P \in \mathbb{R}^{3\times3}$ that projects vectors in $\mathbb{R}^3$ onto the plane W. (i.e, if $v \in \mathbb{R}^3$ is any vector then Pv should be the point in W closest to v.)

(d) Find the point $w^* \in W$ that is closest to $v = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$, either using the projection matrix from above or by other means.

(e) Find a basis for $\mathcal{N}(P)$, the null space of this projection matrix. (Hint: Try this even if you didn't succeed in computing P.)

3. Consider the following least squares problem: Find a function $f(t)$ of the form $f(t) = a + b\sqrt{1+t}$ that gives the best fit to the data

t	y
-1	2
0	1
3	4

(a) Set up the proper over-determined linear system $Ax \approx b$.

(b) Determine the function $f(t)$ by solving the normal equations.

4. The matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, and corresponding eigenvectors

$$r_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a) Let $x_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and define $x_n = A^n x_0 = Ax_{n-1}$. Explain why $\|x_n\| \approx 3\|x_{n-1}\|$ in any vector norm for sufficiently large n, though not necessarily for small n.

(b) Is the same true for *any* starting vector $x_0 \in \mathbb{R}^2$?

5. Let $A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) Determine the eigenspace associated with each eigenvalue.

6. Are each of the following statements true or false? Justify your answers by showing why it should be true or providing a counterexample.

(a) Let $x, y \in \mathbb{R}^n$. If $x + y$ is orthogonal to $x - y$ then $\|x\|_2 = \|y\|_2$.

(b) Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$. Then $\mathcal{N}(AB) = \mathcal{N}(B)$. (Where $\mathcal{N}$ denotes the null space of a matrix.)