Preliminary Exam in Linear Algebra
January 10, 2002

One page front and back of notes allowed. Closed book.

1. Consider the linear system

$$\begin{aligned} 6x_1 - 2x_2 &= 4 \\ 3x_1 + 5x_2 + x_3 &= 8 \\ -2x_2 + 4x_3 &= -2 \end{aligned}$$

   (a) Set this up in the form $Ax = b$

   (b) Factor this into $PA = LU$, where P is a permutation matrix, L is unit lower triangular, and U is upper triangular.

   (c) Use these factors and do both forward and backward substitution to solve the system.

   (d) Did you have to permute? Why or why not?

2. Consider the following least squares problem: Find a function $b(t) = c + d\sqrt{(1+t)}$ that gives the best fit to the data (t, b): (-1,2), (0,5), and (3,8).

   (a) Set this up as an over-determined linear system $Ax \approx b$.

   (b) Determine the function $b(t)$ by solving the normal equations.

   (c) Compute the residual associated with your answer.

   (d) Use the Gram-Schmidt procedure to show that the orthonormal vectors $q_1 = [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]^T$ and $q_2 = [-1/\sqrt{2}, 0, 1/\sqrt{2}]^T$ form an orthonormal basis for the column space of the matrix A above.

   (e) Find an orthonormal projector that projects the vector $[2, 5, 8]^T$ onto the column space of A.

   (f) What do you get when you apply this projector to the vector $[2, 5, 8]^T$? Does this help explain the residual you calculated earlier?

3. The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$, and corresponding eigenvectors $v_1 = [1, 1]^T$ and $v_2 = [1, -1]^T$.

   (a) Let $x_0 = [2, 3]^T$ and define $x_n = (A^{-1})^n x_0$. Explain why $||x_n|| \approx ||x_{n-1}||$ in any vector norm after a few iterations (n around 3 or 4).

   (b) Give an initial guess x_0 for which this would not hold. For your initial guess, can you make any statement that relates the norms of x_n and x_{n-1}?

4. Let $A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$

   (a) What are the eigenvalues of A?

   (b) Which eigenvalue(s) have two linearly independent eigenvectors?

(c) Which eigenvalue(s) have one eigenvector and one principal vector? Find both of these vectors.

5. Are each of the following statements true or false? Justify your answers by showing why it should be true or providing a counterexample.

   (a) Let A be a 2×2 matrix whose determinant is zero. Then, there is no 2×1 vector b so that the equation $Ax = b$ has any solution.

   (b) The matrix in problem 4 is diagonalizable.

   (c) Let $A_{m \times n}$ and $B_{n \times m}$ be real matrices. Then the null space of matrix BA is the same as the null space of matrix B.

   (d) A permutation matrix interchanges the rows (columns) of a matrix when it pre (post) multiplies the matrix.

   (e) Let Q be a matrix with orthonormal columns. Then Qx has the same 2-norm as vector x.