Winter 2003

Preliminary Exam in Linear Algebra

January 15, 2003

1. Let

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 3 \end{bmatrix}.$$

   (a) Let A be the matrix with columns v_1, v_2, and v_3. Consider the linear system $Ax = b$ with

$$b = \begin{bmatrix} 1 \\ 6 \\ 4 \\ 9 \end{bmatrix}.$$

   How many solutions does this system have? If it has at least one solution, determine all solutions.

   (b) Determine a basis for the null space of A.

   (c) Are the vectors v_1, v_2, and v_3 linearly independent? Justify your answer.

2. Consider the following least squares problem: Find a function $f(t)$ of the form $f(t) = a + bt$ that gives the best fit to the data

t	y
1	0
2	1
3	0
4	1

   (a) Set up the proper over-determined linear system $Ax \approx b$.

   (b) Determine the function $f(t)$ by solving the normal equations.

   (c) The matrix A can be factored as $A = QR$ with

$$Q = \frac{1}{2}\begin{bmatrix} 1 & -3/\sqrt{5} \\ 1 & -1/\sqrt{5} \\ 1 & 1/\sqrt{5} \\ 1 & 3/\sqrt{5} \end{bmatrix}, \qquad R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}.$$

   Explain how the Gram-Schmidt process can be used to determine this QR factorization, and determine the r_{ij}.

   (d) Use this QR factorization to solve the least-squares problem. (If you didn't succeed in computing the r_{ij}, at least explain the process.)

3. Let $V = \{x \in \mathbb{R}^4 : \ x_2 = x_3 = x_4\}$, a 2-dimensional subspace of $\mathbb{R}^4$.

   (a) Determine an orthonormal basis $\{q_1, q_2\}$ for V.

   (b) Let Q be the matrix with columns q_1 and q_2 and explain why $P = QQ^T$ is the projection matrix that orthogonally projects any vector in $\mathbb{R}^4$ onto V.

   (c) Determine the eigenvalues of the 4×4 matrix P and the linear space of eigenvectors corresponding to each. (Hint: think about the fact that P is a projection matrix.)

4. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   (a) Determine the eigenvalues and eigenvectors of A. Hint: it is block diagonal.

   (b) Can A be diagonalized by a similarity transformation? If so, determine the symmetrizing matrix and resulting diagonal matrix. If not, explain why not.

   (c) Let

$$v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

   Express v as a linear combination of the eigenvectors of A. Using this, determine $w = A^{1000}v$.

   (d) The matrix A is positive definite, and can be factored as $A = W^T W$. Determine an upper triangular matrix W that works in this "Cholesky decomposition". (Hint: Use Gauss elimination to first determine a decomposition $A = LDL^T$.)