Preliminary Exam in Linear Algebra
January, 2004

1. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 3 & 0 & 6 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

   (a) Determine the permutation matrix P so that

$$PAP^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

   (b) Determine the eigenvalues of the matrix A and a basis for each eigenspace.

2. (a) Let $A \in \mathbb{R}^{m \times n}$. Show that $\mathcal{N}(A^T)$ is orthogonal to $\mathcal{R}(A)$ (where $\mathcal{N}$ and $\mathcal{R}$ denote the null space and range).

   (b) Suppose $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = n$. Show that $\text{rank}(A^T A)$ =n. Hint: Use part (a).

3. Let

$$A = \begin{bmatrix} -1 & -4 \\ 2 & 5 \\ 2 & 2 \end{bmatrix}.$$

   (a) Let $S = \{b \in \mathbb{R}^3 : \text{the least squares solution to } Ax = b \text{ is } x = 0\}$. Determine a basis for S.

   (b) Use Gram-Schmidt to determine the QR factorization of the matrix A with $Q \in \mathbb{R}^{3 \times 2}$ and $R \in \mathbb{R}^{2 \times 2}$.

4. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 1 \\ 3 & 0 & 5 \end{bmatrix}.$$

   (a) Determine the LU factorization of the matrix A (without pivoting).

   (b) Determine *all* solutions to the linear system $Ax = b$ for $b = [5, 8, 7]^T$.

5. Find the point on the line $y = 2x + 1$ that is closest to the point $(1, 5)$.
Note: this line does not go through the origin.

6. True or false? **Justify your answers!**

(a) The sum of two rank 1 matrices always has rank 2.

(b) Let $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $\{x \in \mathbb{R}^2 : \ b^T x = 3\}$ is a linear subspace of $\mathbb{R}^2$.

(c) If P is a projection matrix then so is $I - P$.

(d) If A is a nonsingular square matrix and $x^T A y = 0$ then at least one of the vectors x or y is the zero vector.