Preliminary Exam

Linear Algebra

Winter 2008

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1. (**20 points, 5 points each**)

   (a) Let A be a 5×5 matrix. Suppose that you know that $\text{rank}(A^2) < 5$. What can you say about $\text{rank}(A)$?

   (b) Write down

   $$\det \begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix}$$

   as a polynomial in x, either factored or expanded.

   (c) What are the singular values of a $1 \times N$ matrix? What is the pseudo-inverse of a $1 \times N$ matrix?

   (d) Write down the formula for the least-squares solution of the system $Ax = b$. Show that this formula reduces to $x = A^{-1}b$ if A is square and invertible.

2. (**20 points**) The 5×5 matrix A is has as its only eigenvalues -1, 2 and 1. All its eigenvectors are:

   $$\lambda = -1 : \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 3 \\ 1 \\ 0 \end{pmatrix},$$

   $$\lambda = 1 : \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ -1 \end{pmatrix},$$

   $$\lambda = 2 : \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}.$$

   Write down all possible Jordan forms of A (with eigenvalues occuring on the diagonal in the order -1, 2, 1).

3. (**20 points**) Consider the transformation L, which maps all 2×2 matrices A to $L(A) = A + A^T$.

   (a) Show that L is a linear transformation.

(b) Find a basis for $\ker(L)$. What is $\dim \ker(L)$? Here ker denotes the null space.

(c) Find a basis for $\mathrm{Im}(L)$. What is $\dim \mathrm{Im}(L)$? Here Im denotes the range.

4. (**20 points** The Cayley-Hamilton theorem) This theorem states that any square matrix satisfies its own characteristic equation. Prove it, following the steps below.

   (a) Prove that the theorem holds for square matrices that may be diagonalized.

   (b) Prove that the theorem holds for Jordan blocks, *i.e.*, matrices of the form $\lambda I + J$, where $\lambda \in \mathbb{C}$, I is the identity matrix and J is the matrix with zeros everywhere, except immediately above the diagonal, where it has 1's.

   (c) Prove the theorem for all square matrices.

5. (**20 points**) Let A and B be matrices of size 3×2 and 2×3 respectively. Suppose that their product is

$$AB = \begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}.$$

   What is BA? It may be useful to calculate $(AB)^2$ and compare the result with AB.