Ordinary Differential Equations 2006

(1) [10pt] Consider the linear homogeneous ordinary differential equation:

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0.$$

Suppose we know one nonzero solution $y(t)$. Using the method of reduction of order to derive the second-order ODE for the other two solutions.

(2) [25pt] Find the general solution of the ODEs:

(a) $y' = -y\,(2x - ye^y)^{-1}$,

(b) $y'' + 4y' + 4y = e^{-x}$ by using Laplace transform.

(3) [15pt] Solve the boundary value problem

$$y'' + (f(x)y)' = 0, \quad y(0) = 1, \quad y'(1) = 0$$

in which $f(1) = 0$.

(4) [25pt] Consider the nonlinear, planar ODE system

$$\begin{aligned} \dot{x} &= bx + cy \\ \dot{y} &= -ax - by - y^3 \end{aligned}$$

in which $b > 0$.

(a) Find the equilibrium points of the system, and obtain the linearly approximated equation near the equilibrium points.

(b) Determine the linear stability of the equilibrium points according to $ac - b^2 > 0$ and < 0.

(c) Classify the equilibrium points of the nonlinear system according to $ac - b^2 > 0$ and < 0.

(5) [25pt] Consider the nonlinear, planar ODE system

$$\begin{aligned} \dot{x} &= bx + \frac{b^2}{a}y \\ \dot{y} &= -ax - by - y^3 \end{aligned}$$

where $a, b > 0$. [Note: This problem is in fact the case of $ac - b^2 = 0$ for problem (4). However, the two problems can be solved independent of each other.]

(a) Linearize the equation near the equilibrium point $(0, 0)$. Find out the two eigenvalues for the linear system.

(b) Introducing the new variables

$$w = y, \qquad z = ax + by.$$

Obtain the ODEs for (w, z).

(c) For the resulting equation in (b), assuming near the origin we can approximate $z \pm w^3$ by z, obtain an approximated system of equations.

(d) Use the approximated equations in (c) to determine the phase portrait in wz-plane near the equilibrium. Provide a sketch for the phase portrait. What is the stability of the equilibrium?