Preliminary Examination
Ordinary Differential Equations

Spring 2003

1. Which of the following functions can be transformed using the Laplace Transform? In addition to your answer state your reason(s) for same. (10 points)

   (a) $A_0 e^{at^2}$, A_0 and a positive constants

   (b) $F_0 \cos(\omega t)$, F_0 a positive constant

   (c) $C_0 t^2$, C_0 a positive constant

   (d) $D_0 t^{-1/2}$, D_0 a positive constant

   (e) $+f_0$, $0 \leq t < 1$; $-f_0$, $1 \leq t < 2$; $+f_0$, $2 \leq t < 3$; 0, $t \geq 3$

2. Determine the transforms for both (d) and (e) in (1) if they exist. (10 points)

3. Consider the equation $xy'' + xy' + 2y = 0$. Determine whether or not there is a singularity and, if so, what kind? Then, find a solution by whatever means you feel appropriate. (10 points)

4. Solve the initial-value problem as defined by:

   $$\ddot{x} + \dot{x} + x = X_0; \quad x(0) = -1, \quad \dot{x}(0) = 2; \; X_0 \text{ a positive constant}$$

   Sketch your result. (20 points)

5. Consider a chemostat where a substrate (such as glucose) is injected into a system. Let S = Substrate and B = Bacteria Content. This system can be modeled as a coupled pair of equations. (30 points)

   $$\frac{dS}{dt} = k(S_T - S) - \frac{m}{Y} SB$$
   $$\frac{dB}{dt} = mSB - kB$$

   where k, m, Y, and S_T are all constants and are positive.

(a) With definitions for NONDIMENSIONAL quantities as

$$\tau = t/\tau_0, \quad \overline{S} = S/S_0, \quad \overline{B} = B/B_0$$

determine τ_0, S_0, and B_0 such that the nondimensional system is

$$\frac{d\overline{S}}{d\tau} = (1 - \overline{S}) - \alpha\overline{S}\,\overline{B}$$
$$\frac{d\overline{B}}{d\tau} = \alpha\overline{S}\,\overline{B} - \overline{B},$$

with α a nondimensional constant.

(b) What is the definition of α in terms of the original parameters, and what does it correspond to physically?

(c) Determine the full dynamics (i.e. solve) for the system.

(d) What happens as $t \to \infty$?

(e) Find equilibria, if any exist, and discuss their physical meaning.

(f) Determine the stability of any equilibria.

(g) Plot any phase plane behavior. (You do not need to find an explicit solution in the phase plane.)

6. Solve the following system via matrix method. (20 points)

$$\dot{\mathbf{X}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{X}$$