2007−2008 ODES Prelim

Show your work to earn credit ! Do not use a calculator.
($y' = dy/dx$; $\dot{x} = dx/dt$.)

1. Graph the solution curves of

$$y' = y^2 - 2y - 3$$

using the direction field for the differential equation. Be sure to identify the zero-slope isoclines (equilibria), regions of increase and decrease, inflection points, and the concavity of your solutions.

2. Solve the first-order equation

$$\frac{dy}{dx} + 2e^x y = 2e^x \sqrt{y} .$$

3. Solve the nonhomogeneous equation

$$x^2 y'' - 3xy' + 3y = 2x^3 - 2x .$$

4. Solve the initial-value problem

$$\frac{d^3x}{dt^3} - 6\frac{d^2x}{dt^2} + 11\frac{dx}{dt} - 6x = e^{4t}$$

$$x(0) = 0, \ \dot{x}(0) = 0, \ \ddot{x}(0) = 1$$

using the Laplace transform.

5. Consider the differential equation

$$2x^2 y'' + x(1+2x)y' - y = 0 .$$

Verify that the origin is a regular singular point. Calculate two linearly independent series solutions about $x = 0$. (Three terms for each series will suffice.)

6. Determine the general solution of the linear, nonhomogeneous system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

using matrix methods.

7. Consider the planar system

$$\frac{dx}{dt} = x - y , \quad \frac{dy}{dt} = 1 - xy .$$

(The variables x and y may be positive or negative.) Determine the nature and stability of each equilibrium. Draw the phase portrait for this system.