Winter 2003

Ordinary Differential Equations
Preliminary Examination

January 2003

1. Solve the following differential equation:

$$x^2y'' + xy' - y = 0.$$

2. Given a solution, $y_1(x) = x$, determine the complete solution for:

$$x^2y'' + 3xy' - 3y = 0; \quad x > 0$$

using the method of reduction of order.

3. Determine the complete solution for $g(t)$ for all t where

$$\dot{g} + ag = \begin{cases} G_0 & 0 < t < t_0 \\ 0 & t_0 < t < t_1 \\ G_0 & t > t_1 \end{cases} ; a, G_0 \text{ constants}$$

and make comments on solution if commentary is necessary.

4. Solve the following system of differential equations:

$$\dot{\mathbf{x}} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

   (a) Determine the homogeneous solutions.

   (b) Using the method of variation of coefficients, solve for the particular solutions.

5. Determine whether the specified value of t is a regular singular point of the differential equation in each example:

   (a) $(\sin t)\ddot{x} + (\cos t)\dot{x} + \frac{1}{t}x = 0;\ t = 0.$

   (b) $(e^t - 1)\ddot{x} + e^t\dot{x} + x = 0'\ t = 0.$

   (c) $t(t-2)^2\ddot{x} + t\dot{x} + x = 0;\ t = 0.$

   Then, where appropriate, construct a solution for each differential equation using the method of Frobenius.

6. A variation to the problem that describes the motion of a pendulum is to allow for rotation about a vertical axis. When this is done, the governing equation will be:

$$\frac{d^2\theta}{dt^2} + (1 - \lambda\cos\theta)\sin\theta = 0$$

where λ is a nondimensional parameter that measures the effect of the rotation compared to the force of gravity.

(a) Determine the effect of λ in the system; compare your results to $\lambda = 0$.

(b) Determine a solution.

(c) Determine equilibria, if such exists.

(d) Illustrate all reasoning using the phase plane.