Differential Equations Prelim
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1. Consider the differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

with the two linearly independent homogeneous solutions y_1 and y_2.

(a) Derive the general solution

(b) Abel's theorem states that the Wronskian is given by

$$W[y_1, y_2] = c\exp\left(-\int p(t)dt\right)$$

using this fact, show that if given y_1, then y_2 can be obtained from a first order differential equation (simplify as much as possible)

2. Consider the Euler differential equation

$$x^2y'' + \alpha xy' + \beta y = 0$$

(a) find the two linearly independent solutions (i.e. generate the indicial equation)

(b) for the case of double roots, use reduction of order to generate the second solution.

3. Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$$

(a) solve the system using undetermined coefficients

(b) solve the sytem using variation of parameters

4. Solve the following differential equation using Laplace transforms

$$y'' + 4y = 2g(t) + 13\delta(t-\pi) \quad y(0) = 10,\ y'(0) = 3$$

where

$$g(t) = \begin{cases} t & 0 < t < 1 \\ 1 & t \geq 1 \end{cases}$$

(a) express $g(t)$ in terms of a Heaviside function

(b) solve the differential equation.

5. Consider the nonlinear system of equations

$$x' = (2+x)(y-x)$$
$$y' = (4-x)(y+x)$$

(a) find all equilibrium points of the system

(b) classify each equilibrium point

(c) sketch the phase-plane dynamics