Ordinary Differential Equations 2005

(1) [25pt] Consider the linear ODE with constant coefficients

$$ay'' + by' + cy = 0 \tag{1}$$

in which $a \neq 0$.
(a) If $b^2 - 4ac = 0$, then the corresponding characteristic polynomial has two equal roots, say λ. Use the method of reduction of order to obtain a second solution that is linearly independent of $e^{\lambda t}$.
(b) Alternatively, you should be able to obtain the two linearly independent solutions in (a) via a limiting process by assuming $b^2 - 4ac \neq 0$ and obtaining two distinct solutions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, and then carrying out $(\lambda_1 - \lambda_2) \to 0$. You are asked to carry out this approach and show that it yields the same results. [Hint: consider a specific solution to (1) with $y(0) = y_0$ and $y'(0) = y_1$. The two methods should yield the same solution since the solution depends continuously on the parameters of the differential equation.]

(2) [25pt] Consider the pair of linear first order ODEs:

$$\begin{cases} x' &= \alpha x + \beta y \\ y' &= \gamma x + \delta y \end{cases} \tag{2}$$

(a) Show that the two linearly independent solutions $\binom{x_1}{y_1}(t)$ and $\binom{x_2}{y_2}(t)$ are orthogonal in the phase plane if and only if $\beta = \gamma$.
(b) Apply this result to the second order equation

$$y'' + by' + cy = 0 \tag{3}$$

and show that the above result indicates that the product of the two roots of the characteristic equation of (3) is -1.

(3) [25pt] Find the general solution of the ODEs:
(a) $x^2 y'' - 3xy' + 4y = \ln x$
(b) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 2e^{-y}$ [Hint: set $\frac{dy}{dx} = p$ and then solve $p(y)$ from the pair of first-order nonlinear ODEs.]

(4) [25pt] Consider the following ODE from classic mechanics (Newton's law):

$$m\frac{d^2x}{dt} = F(x) - \eta\frac{dx}{dt}, \quad (m \neq 0) \tag{4}$$

where $F(x) = -\frac{dU(x)}{dx}$ is the force of a potential function $U(x)$, and $-\eta\frac{dx}{dt}$ is due to frictional force. The total energy of the system at time t, $E(t)$, is the sum of the kinetic energy $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2$ and $U(x)$.
(a) What is $\frac{dE(t)}{dt}$? Simplify as much as possible and explain your finding.
(b) If $\eta = 0$, show that

$$t - a = \int_b^x \sqrt{\frac{m}{2(c - U(x))}}dx$$

where a, b, and c are constants. What is the meaning of each of these constants?
What happens for an x^* where $U(x^*) = c$?