

Analysis of The Discrete Nagumo Equation: Modeling Traveling Waves in Myelinated Neurons

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June 1, 2007
AMATH 504

1 Introduction

In class we have analyzed the Nagumo Equation as a model for the spread of invading organisms. We have shown that the equation supports a traveling wave solution provided that population densities exceed some threshold level. Another biological system that exhibits traveling wave and threshold dynamics is the transmembrane potential of a neuron. In this paper, we consider the Nagumo Equation in the context of mathematical neuroscience.

In Section 2 we derive the Nagumo Equation as a model for transmembrane voltage in neurons. In Section 3, we consider a discrete analogue to the Nagumo Equation as a model for myelinated neurons. We discuss the significant differences between the behavior of traveling wave solutions for these two related equations. Finally, in Section 4, we apply these results to models of external electrical stimulation of neurons, focusing in particular on applications to cochlear implants.

2 Mathematical Model of Neurons

We first derive the Cable Equation as a model for current flow through a spatially extended neuron. The Cable Equation was originally derived by Lord Kelvin in 1855 and later applied to neurons by Hodgkin and Rushton in 1946 and Wilfrid Rall in the late 1950's and 1960's. We then formulate the continuous and discrete Nagumo Equations within the framework of the Cable Equation. Much of the following relies on Chapters 8 and 9 of [3]. There the Nagumo Equation is referred to as the Bistable Equation.

2.1 Anatomy of a Neuron

There are an estimated 10^{12} nerve cells, or neurons, in the human body [2]. These cells are responsible for transmitting signals to and from the central nervous system. The structure of a neuron is tailored to its particular function, but several features are common to all neurons. Figure 1 depicts a typical neuron. The neuron consists of branching ends (dendrites and presynaptic terminals), a cell body (the soma), and the axon which connects the cell body to the presynaptic terminals.

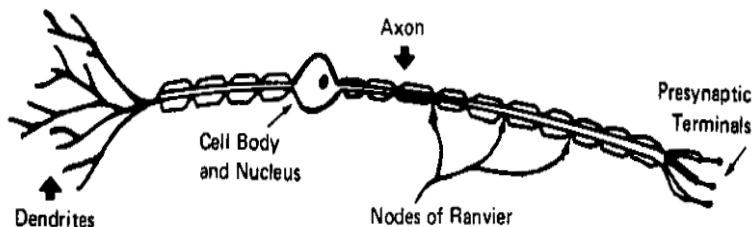


Figure 1: Depiction of a neuron

The signal that is transmitted from neuron to neuron is known as an action potential or spike. In response to chemical or electrical stimuli, a neuron can be excited. The

result of excitation is a sharp increase in the neuron's transmembrane electrical potential at the site of spike initiation. This spike in voltage then propagates down the axon to the presynaptic terminals.

Nerve cells can be divided into two major classes depending on the structure of the axon. The majority of nerve cells are myelinated. As depicted in Figure 1, the axon of a myelinated neuron is covered by sheaths of lipid material (called myelin) with periodic gaps in the myelin called nodes of Ranvier. As the name suggests, nonmyelinated neurons have axons with no lipid covering about the axon membrane.

Action potential propagate very differently for myelinated and nonmyelinated neurons. In a nonmyelinated neuron, current can flow continuously along the axon as if it were an electrical wire. A reaction-diffusion equation such as the Nagumo Equation may be appropriate for modeling action potential propagation in nonmyelinated neurons. In contrast, action potentials in myelinated neurons propagate by jumping from node to node. This mode of propagation is called saltatory and may be better modeled in a discrete spatial domain.

2.2 The Cable Equation

We now derive a model for transmembrane potential in the axon of nonmyelinated neurons. The Nagumo Equation will arise naturally out of this derivation. Neglect spatial variations in the neuronal structure and assume that a neuron can be approximated as a long cylindrical membrane with no radial or angular variation in electrical potential. Then the electrical field is completely characterized by the longitudinal coordinate along the membrane, denoted x . The extracellular potential is denoted $V_e(x)$ and the intracellular potential is denoted $V_i(x)$. Now consider a small section of the neuron of length Δx .

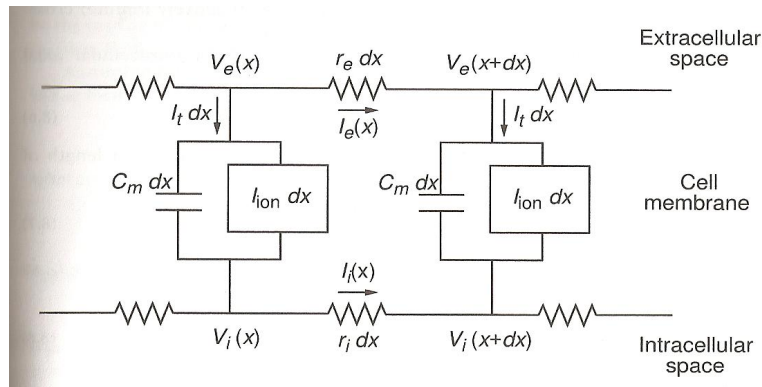


Figure 2: Circuit diagram for the cable model of a neuron, image from [3]. Note dx in the figure corresponds to Δx in the text.

See Figure 2 for a circuit diagram of the Δx section of the neuron. The current flow along the membrane has intracellular and extracellular components, denoted I_i and I_e , that are assumed to follow Ohm's Law:

$$V_i(x + \Delta x) - V_i(x) = -I_i(x)r_i\Delta x \quad (2.1)$$

$$V_e(x + \Delta x) - V_e(x) = -I_e(x)r_e\Delta x, \quad (2.2)$$

where r_i and r_e are the resistivities per unit length of the intracellular and extracellular media, respectively, and the negative sign is by convention.

In response to the axial current flow, Kirchoff's Law requires that there must be some transmembrane current, $I_t(x)$, satisfying

$$I_t(x)\Delta x = I_i(x) - I_i(x + \Delta x) = -I_e(x) + I_e(x + \Delta x). \quad (2.3)$$

At this stage, one typically allows $\Delta x \rightarrow 0$. Taking this limit is reasonable when modeling the axons of nonmyelinated neurons since there is relatively little variation in the physical and electrical properties of the axon. Thus

$$I_t(x) = -\frac{\partial I_i}{\partial x} = \frac{\partial I_e}{\partial x}. \quad (2.4)$$

In the absence of an applied current, the total axial current is the constant $I_T = I_i + I_e$, then from the Ohm's Law equations 2.1 and 2.2:

$$-I_T = \frac{1}{r_i} \frac{\partial V_i}{\partial x} + \frac{1}{r_e} \frac{\partial V_e}{\partial x}. \quad (2.5)$$

Now define the transmembrane potential as $V = V_i - V_e$, and use it to replace V_e in the above equation to get

$$\frac{1}{r_i} \frac{\partial V_i}{\partial x} = -\frac{r_e}{r_i + r_e} I_T + \frac{1}{r_i + r_e} \frac{\partial V}{\partial x}. \quad (2.6)$$

Note that the left hand side is $I_i(x)$ if we take the limit as $\Delta x \rightarrow 0$ in Equation 2.1. If we take an additional spatial derivative, then the left hand side becomes I_t by Equation 2.4. Furthermore, I_T is constant and so it becomes 0 after the derivative. Thus

$$I_t(x) = \frac{1}{\partial x} \left(\frac{1}{r_i + r_e} \frac{\partial V}{\partial x} \right). \quad (2.7)$$

We now derive an alternative expression for $I_t(x)$ based on the chemical gradients across the cell membrane. There are typically large gradients in K^+ , Na^+ , and other ions that induce a flow of current (since they are charged particles) as they diffuse across the cell membrane. The exact dynamics of these currents will not be explored here, but we lump their effects together in a term denoted by $I_{ion}(x)$. In addition to the ion current, there is also a current driven by the capacitance of the cell membrane. A basic fact from the theory of electrical circuits is that capacitive current is proportional to the time derivative of voltage. Thus the equation relating $I_t(x)$ to the ion and capacitive currents is

$$I_t(x) = p \left(C_m \frac{\partial V}{\partial t} + I_{ion} \right) \quad (2.8)$$

where p is the perimeter of the axon and C_m is the capacitance per unit area of the membrane.

One can rescale (see [3] for details) and equate these two expressions for I_t to recover the Cable Equation:

$$\frac{\partial V}{\partial T} = \frac{\partial^2 V}{\partial X^2} + f(V, T) \quad (2.9)$$

where T , X are rescaled time and space variables and $f(V, T)$ is a function of the ionic current. The choice of $f(V, T)$ depends on the model, but if we let it be a cubic polynomial of the form

$$f(V) = V(V - 1)(\alpha - V), \text{ where } 0 < \alpha < 1, \quad (2.10)$$

Then we recover the familiar Nagumo Equation. Perhaps the most widely used model for ion channel dynamics is the famous Hodgkin-Huxley equations. The choice of $f(V, T)$ as the above cubic form can be motivated by a phase plane analysis of the slow dynamics of the Hodgkin-Huxley Equations [9].

2.3 Compartmental Models for Myelinated Neurons

Deriving the Cable Equation as a model for voltage in a nonmyelinated neuron required taking the limit as Δx tends to 0. This assumes that the electrical potential along the neuron varied smoothly in space (to be precise, we required $V \in C^2(\mathbb{R})$). This assumption is reasonable for nonmyelinated neurons but is not ideal for modeling myelinated neurons because of the great difference in electrical conductance properties of the myelinated sheath compared to the nodes of Ranvier.

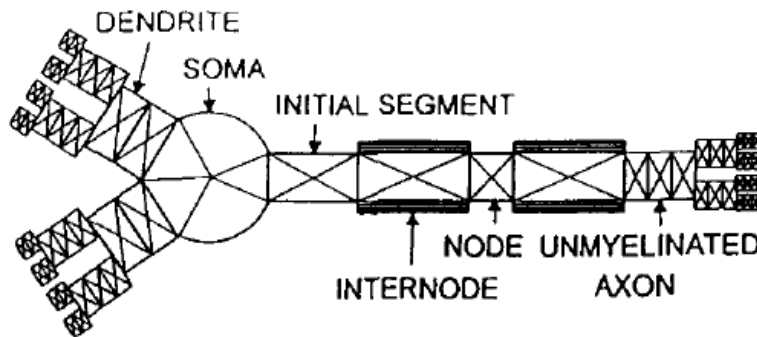


Figure 3: Schematic of a compartmental model of a neuron [7]

Rather than allowing $\Delta x \rightarrow 0$, myelinated neurons are frequently treated with compartmental models. Compartmental models partition the neuron into different sections, see Figure 3. For example, a myelinated area would be distinct from an adjacent node of Ranvier. Additional compartments could be included to model the soma and the branched structures of the dendrites and presynaptic terminals. Within each compart-

ment, the transmembrane voltage is modeled with a system of ODEs, for example the Hodgkin-Huxley equations. Voltage does not diffuse longitudinally in accordance with the Cable Equation; instead nearby compartments are coupled together. The resulting system of coupled equations is a differential (in time) difference (in space) equation.

We now derive a discrete analogue to the Cable Equation that is more appropriate for applications to myelinated neurons. Consider a myelinated neuron axon and neglect all other structures such as the cell body and dendrites. Take the myelinated sheaths to be perfect conductors and only model the voltage at the nodes of Ranvier. The equation relating capacitance to voltage (Equation 2.8) governs the transmembrane potential dynamics at each node. Let $I_{i,n}^{(in)}$ and $I_{i,n}^{(out)}$ denote the axial intramembrane current flowing into and out from the n^{th} node, respectively, and similarly denote $I_{e,n}^{(in)}$ and $I_{e,n}^{(out)}$ for extracellular currents. Then Kirchoff's Law requires both

$$\begin{aligned} I_{t,n} &= I_{i,n}^{(in)} - I_{i,n}^{(out)} \\ I_{t,n} &= -I_{e,n}^{(in)} + I_{e,n}^{(out)}. \end{aligned} \quad (2.11)$$

Each of the terms on the right side can be rewritten using Ohm's Law. For instance,

$$I_{i,n}^{(in)} = \frac{1}{r_i \Delta x_r} (-V_{i,n} + V_{i,n-1}) \quad (2.12)$$

where Δx_r is the length of the node. Replacing the other intracellular and extracellular current terms with Ohm's Law relations yields

$$\begin{aligned} I_{t,n} &= \frac{1}{r_i \Delta x_r} (V_{i,n+1} - 2V_{i,n} + V_{i,n-1}) \\ I_{t,n} &= \frac{1}{r_e \Delta x_r} (V_{e,n+1} - 2V_{e,n} + V_{e,n-1}). \end{aligned} \quad (2.13)$$

Now define the transmembrane electrical potential at the n^{th} node to be $V_n = V_{i,n} - V_{e,n}$. Then the above results can be combined with the equation relating capacitance to voltage (Equation 2.8):

$$p \Delta x_s \left(C_m \frac{\partial V}{\partial t} + I_{ion} \right) = \frac{1}{(r_i + r_e) \Delta x_r} (V_{n+1} - 2V_n + V_{n-1}), \quad (2.14)$$

where Δx_s is the length of the myelinated sheath separating adjacent nodes. Dimensionless time $T = \frac{t}{C_m R_m} = \frac{t}{t_m}$ can be introduced and the equation can be rescaled to recover the Discrete Cable Equation:

$$\frac{dV_n}{dT} = d(V_{n+1} - 2V_n + V_{n-1}) + f(V_n, T), \quad (2.15)$$

The constant $d = \frac{R_m}{\Delta x_r \Delta x_s p (r_i + r_e)}$ is the coupling coefficient and R_m is the membrane resistivity, see [3] for details. As in the continuous case, $f(V_n, T)$ is a function of the ionic currents. It is frequently chosen to be the cubic function in Equation 2.10, in which case we say Equation 2.15 is the Discrete Nagumo Equation.

The coefficients in Equation 2.15 suggest that the Discrete Nagumo Equation is

merely a second difference approximation to the continuous Nagumo Equation 2.9 and thus that they should behave similarly in some limit. As we will see in Section 3, however, the existence and behavior of traveling wave solutions for these two equations can be very different and depends on the coupling coefficient d .

3 Analysis of Discrete Nagumo Equation

Throughout this section we consider the Discrete Nagumo Equation:

$$\frac{dv_n}{dt} = d(v_{n+1} - 2v_n + v_{n-1}) + v_n(v_n - a)(1 - v_n) \quad (3.1)$$

where d is a positive real number and $a \in (0, 1)$.

3.1 Existence of Traveling Waves

The existence of traveling waves for the Nagumo Equation is well-known and in fact we have computed the shape of the traveling wavefront exactly in class. One might expect, therefore, that the Discrete Nagumo Equation also supports a traveling wave solution. Such a traveling wave solution does exist, but the existence proof requires sophisticated methods beyond the scope of this course. The proof is due to Bertram Zinner in 1992 [11].

Theorem 3.1. *There exists some $d^* > 0$ such that for $d > d^*$ the Discrete Nagumo Equation 3.1 admits a solution $v_n(t) = V(n + ct)$, where $c > 0$, $V \in C^1(\mathbb{R}, (0, 1))$, $V(-\infty) = 0$, $V(\infty) = 1$, and $V'(x) > 0$ for all $x \in \mathbb{R}$.*

3.2 Speed of Traveling Waves

Recall that in class we were able to calculate the exact speed of the traveling wave solution for the continuous Nagumo Equation. The wave speed of the continuous version of Equation 3.1 is (as computed in [3])

$$c = \sqrt{\frac{d}{2}}(1 - 2a). \quad (3.2)$$

By Theorem 3.1, the existence of a traveling wave for the discrete equation 3.1 is guaranteed for sufficiently large d . A traveling wavefront will jump from one node to the next at a constant speed. This speed can be computed and compared to the continuous wave. The following treatment follows Chapter 9.3 of [3].

A traveling wave solution to Equation 3.1 satisfies $v_{n+1}(t) = v_n(t - t_d)$ where t_d is the time it takes for the traveling wave to progress one step in the discretized spatial domain. Set $v_n(t) = v(t)$ then Equation 3.1 becomes:

$$\frac{dv}{dt} = d(v(t + t_d) - 2v(t) + v(t - t_d)) + v(t)(v(t) - a)(1 - v(t)). \quad (3.3)$$

Expand $v(t + t_d)$ and $v(t - t_d)$ in Taylor series about t :

$$\begin{cases} v(t + t_d) = v + t_d v_t + \frac{t_d^2}{2} v_{tt} + O(t_d^3) \\ v(t - t_d) = v - t_d v_t + \frac{t_d^2}{2} v_{tt} + O(t_d^3). \end{cases} \quad (3.4)$$

$$\Rightarrow \frac{dv}{dt} = dt_d^2 v_{tt} + v(v - a)(1 - v) + O(t_d^3). \quad (3.5)$$

Recall that in our analysis of the continuous Nagumo Equation, we transformed the Nagumo Equation to a traveling wave coordinate frame which yielded the following ordinary differential equation (Equation 4.95 in the course notes):

$$u'' + cu' + u(u - a)(1 - u) = 0. \quad (3.6)$$

With the exception of the coefficient values, we see that to leading order the equation for the discrete traveling wave equation 3.5 and the continuous Nagumo traveling wave 3.6 are the same. Let $u(t)$ be the traveling wavefront solution for the continuous equation, as derived in class. Then to account for the difference in coefficients, the solution for the discrete traveling wavefront, to leading order, is of the form $u(ct)$ where c is the wave speed for the continuous Nagumo Equation and must satisfy $dt_d^2 = \frac{1}{c^2}$.

The wave speed in the discrete spatial domain, denoted s is the internodal distance $\Delta x_s + \Delta x_r$ divided by the delay time $t_d t_m$. Comparing the discrete and continuous Nagumo wave speeds we have:

$$s = \frac{\Delta x_s + \Delta x_r}{t_d t_m} = \frac{\Delta x_s + \Delta x_r}{t_m} c \sqrt{d}. \quad (3.7)$$

In [3], dimensional units are reintroduced and it is estimated that the discrete traveling wave travels faster than the continuous wave by a factor of $\frac{\Delta x_s + \Delta x_r}{\sqrt{\Delta x_s \Delta x_r}}$. Empirical results indicate that action potentials in myelinated nerve fibers propagate by a factor of $6\sqrt{\frac{d}{10^{-6}m}}$ faster than nonmyelinated nerves, in rough agreement with the first order approximation.

Using model parameter from [7], we estimate that $\Delta x_s \approx 500\mu m$ and $\Delta x_r \approx 2.5\mu m$. Thus the fact that auditory neurons are myelinated increases the speed of action potential propagation by approximately 14. More applications to modeling the auditory nerve will be discussed in Section 4.

3.3 Propagation Failure of Traveling Waves

A surprising difference between the Continuous and Discrete Nagumo Equations is that in the discrete form, traveling waves may fail to propagate. The result was proven by Keener [4] and we provide a sketch of the proof.

Theorem 3.2. *Consider Equation 3.1 with $a \in (0, \frac{1}{2})$. No traveling wave solutions exist provided the coupling coefficient satisfies $d < \frac{a^2}{8}$.*

Note that this does not contradict the existence result in Theorem 3.1 since there the restriction on d was that it be sufficiently large.

To prove the theorem, Keener shows the existence of infinitely many steady state solutions of the Discrete Nagumo Equation. The construction of these steady states uses a theorem of Moser which is similar to constructing a Smale Horseshoe for the map obtained from setting $\frac{dv_n}{dt} = 0$ in Equation 3.1. These methods are beyond the scope of our course, see [10] e.g. But we note that the restriction that d must be sufficiently small to prevent traveling wave solutions is a necessary condition for the existence of these steady states.

Once these solutions have been constructed and shown to be stable, they can be shown to be "blocking solutions" in the sense that they prevent propagation of traveling waves. This is done using a standard comparison result of the form:

Lemma 3.1. *Suppose $\{u_n(t)\}$ and $\{v_n(t)\}$ satisfy*

$$\begin{aligned}\frac{du_n}{dt} &\leq d(u_{n+1} - 2u_n + u_{n-1}) + u_n(u_n - a)(1 - u_n) \\ \frac{dv_n}{dt} &\geq d(v_{n+1} - 2v_n + v_{n-1}) + v_n(v_n - a)(1 - v_n).\end{aligned}$$

If $u_n(t_0) \leq v_n(t_0)$ for all n , then $u_n(t) \leq v_n(t)$ for all $t \geq t_0$.

To conclude the proof, Keener shows that for any initial sequence $\{v_n(0)\}$ and sufficiently small d , the solution can be blocked. In particular, he shows that for sufficiently small d there exist x_0, x_1 in $(0, 1)$ (and satisfying several other conditions that are not enumerated here) such that the intervals $[0, x_0)$ and $(x_1, 1]$ are invariant. Recall that a traveling wavefront solution requires $\lim_{t \rightarrow -\infty} v_n(t) = 0$ and $\lim_{t \rightarrow +\infty} v_n(t) = 1$ or vice versa. Thus the invariance of these intervals is equivalent to propagation failure since if $v_n(0) \in [0, x_0)$ then $v_n(t) \in [0, x_0)$ for all $t > 0$. Similarly if $v_n(0) \in (x_1, 1]$ then $v_n(t) \in (x_1, 1]$ for all $t > 0$. In either case, the possibility of a traveling wave propagation for all time is excluded.

4 Application to Cochlear Implants

Examples of myelinated neurons that are frequently modeled within the discrete Cable Equation framework are auditory nerve fibers. The auditory nerve is a collection of 30,000 to 55,000 nerve fibers, depending on the species, that connect transduction sites in the inner ear to sites in the brain responsible for processing the neural response to auditory stimuli. The great majority of these nerve fibers are so-called Type I afferent neurons that are most responsible for encoding auditory stimuli. These neurons are large and myelinated, and therefore conduct action potentials very fast [1].

In a normal hearing individual, action potentials are generated in the auditory neuron in response to neurotransmitter release at the synapse between innervating neurons and inner hair cells in the cochlea. For individuals with sensorineural hearing loss, the ear does not successfully convert sound stimuli to a neural response in the auditory nerve.

To treat this form of hearing loss, an array of electrodes can be surgically implanted in the cochlea and the auditory nerve can be stimulated with electrical fields produced by the electrode array, see Figure 4.

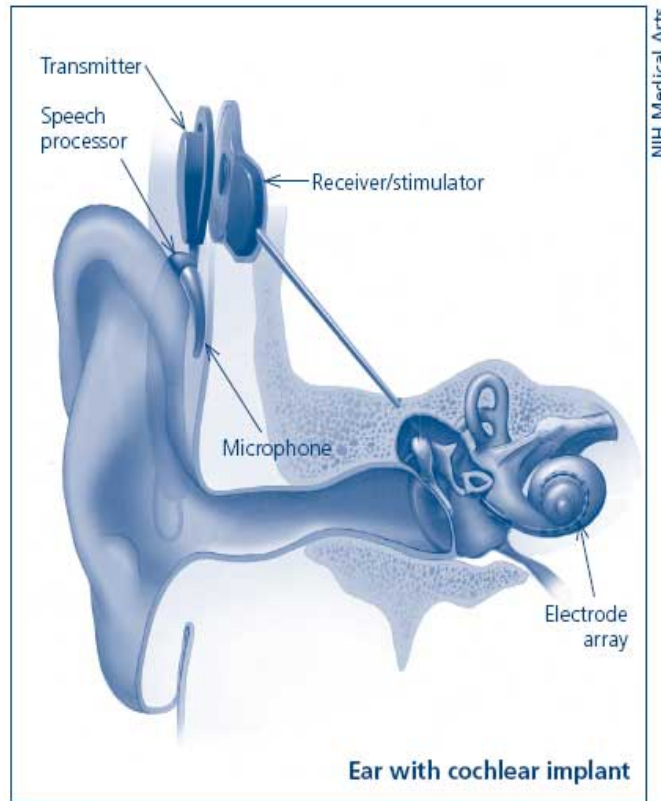


Figure 4: Cochlear Implant, image from [5]

Nearly 100,000 individuals worldwide have received cochlear implants to restore their hearing [5]. Understanding the response of myelinated auditory neurons to applied electrical fields is therefore of considerable interest in further improving cochlear implant performance.

In the remainder of the paper, we consider the above theoretical framework and results to analyze recent approaches to modeling the response of the auditory nerve to external electrical fields.

4.1 Activating Function

A complete simulation of the auditory nerve response to external electrical fields would require modeling the response of thousands of nerve fibers using Hodgkin-Huxley or stochastic ion channel dynamics. Such computations require significant computing resources. A frequently used substitution for Hodgkin-Huxley or other models for action potential generation is the activating function introduced by Rattay to estimate the influence of an applied electrical field on a neuron [6]. The activating function is easily

constructed from the relations derived for the Discrete Cable Equation. Recall Equation 2.8 and the the Ohm's law relation between intramembrane current and intracellular voltage in Equation 2.13. Then the total transmembrane current through the node satisfies

$$p\Delta x_s \left(C_m \frac{\partial V_n}{\partial t} + I_{ion} \right) = \frac{1}{r_i \Delta x_r} (V_{i,n+1} - 2V_{i,n} + V_{i,n-1}). \quad (4.1)$$

Now make the substitution $V_{i,n} = V_n + V_{e,n}$ to get

$$\frac{\partial V_n}{\partial t} = \frac{1}{C_m} \left[\frac{1}{pr_i \Delta x_s \Delta x_r} (V_{n+1} - 2V_n + V_{n-1}) + \frac{1}{pr_i \Delta x_s \Delta x_r} (V_{e,n+1} - 2V_{e,n} + V_{e,n-1}) - I_{ion} \right]. \quad (4.2)$$

Written in this form, it is apparent that the only change to V_n due to an applied electric field is the addition of the terms $V_{e,n+1} - 2V_{e,n} + V_{e,n-1}$. This lead Rattay [6] to characterize the effect of the applied field by the so-called activating function $A_n(t)$ which, after juggling some coefficients, is simply the second difference operator:

$$A_n(t) = \frac{V_{e,n+1} - 2V_{e,n} + V_{e,n-1}}{\Delta x_s^2}. \quad (4.3)$$

In the case of nonmyelinated neurons, the limit $\Delta x_s \rightarrow 0$ is justified so the activating function in that case is

$$A(x, t) = \frac{\partial^2 V_e}{\partial x^2}. \quad (4.4)$$

The activating function is a useful but limited method for approximating where the most intense effects of the external electric field will be along a neuron. Its most obvious limitation is that it neglects the influence of (nonlinear) ion channel dynamics so it is not directly related to the biological mechanisms for action potential generation. Furthermore, Theorem 3.2 indicates that action potentials in a myelinated neuron may fail to propagate as traveling waves. The activating function gives no information about whether propagation failure may occur for certain modes of electrical stimulation. The activating function does not give a complete description of neural dynamics in an applied electric field, but it is relatively simple to compute and therefore is a valuable tool when performing large scale simulations of the auditory neuron.

An example of a study in which the activating function is used is [7]. Rattay uses a compartmental model with parameter values meant to represent an auditory nerve fiber and Hodgkin-Huxley dynamics at the nodes of Ranvier. In some simulations, the neuron is excited by an external electrical field. In others, current is injected directly into the neuron. For neurons in an external electrical field, Ratay computes the activating function and in order to identify what information can be gained from computing the activating potential. In many cases, it appears that peaks of the activating function correspond to the location of action potential generation for a neuron in a uniform electric field. This is a helpful result that allows one to predict the site of action potential initiation in a spatially complicated neuron without having to simulate the full Hodgkin-Huxley dynamics.

A major limitation of the activating function approach is that it cannot predict when an action potential will successfully propagate. Rattay observes that isolated peaks in the activating function may correspond to action potentials that fail to propagate. In such cases it is necessary to consider the contributions of nearby compartments and perhaps the full Hodgkin-Huxley dynamics themselves. While the activating function is, as Rattay puts it, "a powerful tool for obtaining a first impression of the influence of an applied electric or magnetic field on a target neuron," it clearly cannot provide a complete characterization of how to generate propagating action potentials in the auditory nerve in response to the electrical field produced by the electrode array of a cochlear implant. We next consider the case of propagation failure in more detail.

4.2 Propagation Failure

In Section 3.3 we observed that it is possible for a traveling wave solution of the Discrete Nagumo Equation to fail to propagate. Keener observes that in the context of cardiac action potentials, propagation failure can be fatal [4]. While the consequences in the auditory system are not so dire, clearly propagation failure would be detrimental to auditory perception. An example of propagation failure in the auditory nerve is the phenomenon of "anodal block." This is an observed phenomenon whereby a cathodal stimulus of a nerve fiber may prevent conduction along the axon of a nerve fiber [8].

Anodal block in myelinated neurons has been modeled by Rattay [7]. He uses a compartmental model with Hodgkin-Huxley dynamics at the nodes of Ranvier and physiological parameters meant to describe an auditory neuron. In his simulations, a $-0.6mA$ stimulus can block action potential propagation. The site of action potential propagation failure changes depending on the current stimulus. In some simulations there is propagation failure in the compartment representing the soma. In other simulations, the action potential propagates successfully through the soma but then fails at a compartment representing a node of Ranvier in the myelinated axon. See Figure 5.

A subsequent simulation shows propagation failure at the soma for a nonmyelinated axon. Unfortunately, Rattay does not report whether propagation failure within a non-myelinated axon was ever observed.

5 Conclusion

The fact that the Nagumo Equation supports a traveling wave solution with threshold effects makes it a likely candidate for modeling action potential propagation in a neuron. However, the derivation of the Nagumo Equation shows that it is most applicable to spatially homogeneous nerve fibers. Since nerve fibers come in all shapes and sizes, it is necessary to reconsider when to apply the Nagumo Equation.

Action potential propagation in myelinated neurons may be better modeled using a compartmental model approach because of the different biophysical properties of the myelin sheath and the nodes of Ranvier that make up the axon of a myelinated neuron. The compartmental model approach leads to the derivation of a discrete Nagumo

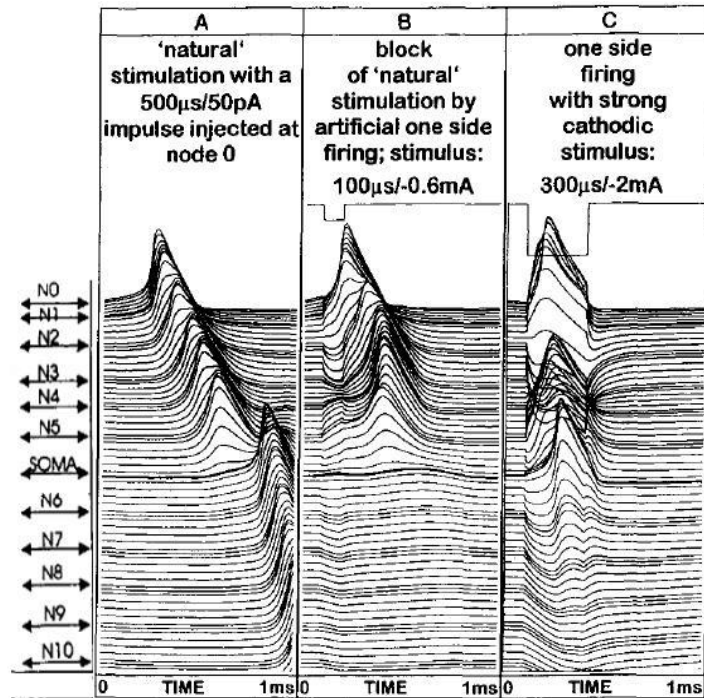


Figure 5: Simulations of propagation failure for a myelinated neuron, image from [7]. Each trajectory corresponds to the transmembrane voltage as a function of time in a given compartment. The location of the compartments representing the nodes of Ranvier and the soma are indicated on the vertical scale at left. Panel A: successful propagation. Panel B: propagation failure at the soma. Panel C: propagation failure at a node of Ranvier (Compartment N6).

Equation. Despite the superficial similarity between the continuous and discrete equations, they exhibit very different dynamics. In particular, traveling waves in the discrete Nagumo Equation travel much faster but may fail to propagate depending on the coupling between adjacent compartments.

Understanding the dynamics of myelinated neurons is of particular interest to researchers striving to improve the performance of cochlear implants. Propagation failure, as predicted by the theoretical analysis of the discrete Nagumo Equation, does seem relevant to this application. In fact, recent computer simulations have exhibited propagation failure in an auditory nerve subject to external electrical stimulation.

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