Wavelet Methods for Differential Equations

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June 8, 2006

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1 Introduction

Differential equations arise in countless mathematical applications. For example, fluid flows, population dynamics, and chemical reactions can all be modeled with systems of differential equations. Ideally the differential equation can be solved exactly using analytic methods, but in many cases (especially in complicated applications) exact solutions can be difficult or impossible to determine. This can be the case even when the existence of such solutions is guaranteed by theoretical results. The development of computer methods to approximate solutions to differential equations is therefore crucial to creating meaningful mathematical models.

The remainder of this paper is organized as follows. In Section 2 some standard numerical methods will be introduced. The applicability of wavelets will be highlighted and familiar properties that make them desirable for approximating solutions to differential equations will be discussed.

In Section 3 approximation schemes that use wavelets will be introduced: one that uses Haar wavelets (Section 3.1), one that uses Daubechies wavelets (Section 3.2), and some modifications to the basic techniques (Section 3.3). Lastly, in Section 4, an application to turbulent flows will be discussed. This application has been chosen because it is a notoriously difficult problem to tackle even with the most sophisticated computational methods. But as we will see, wavelet methods have several features that are well-suited to approximating turbulent flow.

2 Standard Numerical Methods

2.1 Finite Differences

Perhaps the most widely used numerical method for solving differential equations is the finite difference method. Consider a first-order differential equation:

$$x'(t) = f(t)$$

The derivative is formally understood as a limiting operation:

$$x'(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

Numerically, we can not achieve this limit so we must choose an appropriate Δt bearing in mind the tradeoff between higher accuracy and longer computational time as Δt is reduced. For some finite Δt, and under the assumption that $f(t)$ and $x(t)$ are known at the grid points determined by the choice of Δt, the differential equation is converted into a system of equations that can be iterated forward one step at a time:

$$x(t + \Delta t) = x(t) + \Delta t f(t).$$

Thre are many other ways to define the finite difference approximation to the derivative. Finite difference schemes are relatively simple to formulate and to implement making them widely used for solving differential equations. One example, known as the second-order centered difference, is defined as

$$x'(t) \approx \frac{x(t + \Delta t) - x(t - \Delta t)}{2\Delta t}.$$

In Section 3.2 we will discuss a wavelet method using Daubechies wavelets that can be interpreted as a highly accurate centered finite difference scheme.

2.2 Series Expansions

An alternative to finite difference schemes is a family of methods (including spectral and pseudospectral methods) that utilize orthonormal bases to express an unknown function as a sum of known basis functions. Perhaps the most familiar example is the Fourier basis:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \exp(ikt)$$

where a_k are the Fourier coefficients of x. The basic idea in series expansions methods is to truncate this series at some finite wavenumber so that $x(t)$ is approximated by

$$x(t) \approx \sum_{k=-N}^{N} a_k \exp(ikt)$$

where the coefficients can be determined via a discrete Fourier transform. Other terms in the differential equation are similarly transformed and the resulting series can be plugged into the differential equations and the Fourier coefficients can be found, thus defining the approximation of $x(t)$.

Series expansion methods can produce highly accurate results but can be computationally intensive. The creation of the Fast Fourier Transform (FFT) significantly increased the viability of series expansion methods using Fourier basis functions because it enables functions to be efficiently transformed between Fourier space and the physical space in which the function is defined.

Fourier series expansions are frequently used because of the efficiency of the FFT and the desireable property that individual basis functions are eigenfunctions of the derivative operator. That is to say the derivative of a Fourier mode is a scalar multiple of the same mode:

$$\frac{d}{dt}(\exp(ikt)) = ik\exp(ikt).$$

This property simplifies the process of solving for the Fourier coefficients of the approximation function. However, series expansions methods are not restricted to Fourier series. Depending on the application or the problem other basis functions may be sought to approximate the solution of a differential equation.

A more complete introduction to finite difference methods, series expansion methods, and other numerical techniques can be found in many numerical analysis text books (see [11]).

2.3 Why Wavelets?

Wavelet functions (and scaling functions) can also be used as a basis functions to express a given function. In numerical methods, one typically seeks to evaluate a function $x(t)$ at a discrete set of grid points, say $t_1, t_2, \ldots, t_N$. We can interpret the values of $x(t)$ at the grid points as a time series $\mathbf{X}$. Then the the discrete function values can be written in the wavelet basis as $\mathcal{W}^T\mathbf{W}$ where the wavelet coefficients $\mathbf{W}$ can be found via the discrete wavelet transform (DWT). An attractive feature of a wavelet basis is that the computational efficiency of the DWT exceeds even that of the FFT [12]. To solve for the wavelet coefficients requires solving large matrix equations. In addition to the speed of the DWT, computations in a wavelet basis have the added benefit that resulting matrices are frequently sparse matrices which reduces computation time [1].

Another advantage of using a wavelet series expansion is the fact that the construction of wavelets makes them inherently well-suited to decorrelating features on different time and/or spatial scales. Examples where this may be useful are stiff ordinary differential equations ([6], [7]), problems that feature shock formation [2] or boundary layers [10], and turbulent flows [4] because all of these applications have solutions with variations across many different spatial and time scales.

A disadvantage of wavelets (when compared to a Fourier basis) is that the action of differential and integral operators on the basis function can be difficult to determine, depending on the choice of wavelet. Consider the differential equation $x'(t) = f(t)$. Expanding $x(t)$ and $f(t)$ in the wavelet basis this equation can be rewritten as:

$$\frac{d}{dt}[\mathcal{W}^T\mathbf{W_x}] = \mathcal{W}^T\mathbf{W_f}$$

where $\mathbf{W_x}$ denotes the wavelet coefficients of $\mathbf{X}$, and similarly for $f(t)$ at the grid points. If the derivative of the wavelet and scaling functions is known, we could define a differentiation matrix $D \equiv \frac{d}{dt}[\mathcal{W}^T]$ and rewrite the differential equation as:

$$D\mathbf{W_x} = \mathcal{W}^T\mathbf{W_f}$$

and then solve for the wavelet coefficients $\mathbf{W_x}$ to compute an approximation for $x(t)$. Unfortunately, determining D is not straightforward as it is in the Fourier basis. Simply plugging in a wavelet series expansion into a differential equation may not be desirable, but the methods discussed below find ways of overcoming this disadvantage while maintaining high accuracy and computational efficiency.

3 Wavelet Methods

3.1 Haar Wavelets

The Haar wavelet is by far the simplest wavelet so it is natural to attempt to utilize it to solve differential equations. A major shortcoming of Haar wavelets is that they are not differentiable, because of the discontinuous jump in the function value. Thus no differentiation matrix D can be defined. Consequently, a Haar series expansion cannot be differentiated term by term and Haar wavelets therefore seem ill-suited to solving differential equations. In order to avoid this pitfall, Chen and Hsiao [3] cleverly introduce a procedure to expand the derivative of the unknown function, instead of the function itself, in the wavelet basis.

They consider the following first-order linear system of differential equations.

$$x'(t) = Ax(t) + Bu(t), x(0) = x_0$$

where x and u are column vectors and A and B are matrices of appropriate dimension. We will simplify matters by assuming the "system" is a single equation, but the analysis for systems is similar. $x(t)$ is the unknown function we seek to approximate, but we choose to expand $x'(t)$ in a wavelet basis, as well as the function $u(t)$ (some known forcing term). If we seek function values at a set of discrete points in time, then we can write $\mathbf{X'} = \mathcal{W}^T \mathbf{W_{x'}}$ and $\mathbf{U} = \mathcal{W}^T \mathbf{W_u}$. An important contribution of Chen and Hsiao is to derive the integration matrix for Haar wavelets. It has a relatively simple structure due to the fact that Haar wavelets are piecewise constant. They define the integration operation matrix $P\mathcal{W}^T = \int_0^t \mathcal{W}^T(s)ds$. Then from $x(t) = \int_0^t x'(s)ds + x_0$ we can write

$$\mathbf{X} = \int_0^t \mathcal{W}^T \mathbf{W_{x'}} ds + \mathbf{X_0} = (\int_0^t \mathcal{W}^T ds)\mathbf{W_{x'}} + \mathbf{X_0} = P\mathcal{W}^T \mathbf{W_{x'}} + \mathbf{X_0}.$$

And now the differential equation can be reformulated as

$$\mathcal{W}^T \mathbf{W_{x'}} = A(P\mathcal{W}^T \mathbf{W_{x'}} + \mathbf{X_0}) + B\mathcal{W}^T \mathbf{W_u}$$

The unknown here are the wavelet coefficients $\mathbf{W_{x'}}$. Fortunately they can be solved for from this equation, and plugged into the previous equation to determine $\mathbf{X}$, the approximation for $x(t)$.

This basic procedure has been extended to nonlinear equations [7]. It has also been refined and applied to several test problems ([3],[6],[7],[10]). It has been shown to produce fast and accurate results, even for differential equations that can be problematic to solve numerically.

Consider the following example, as solved by Lepik [10].

$$-\epsilon x''(t) + x'(t) = 1, t \in [0, 1]$$
$$x(0) = x(1) = 0$$

This is a second-order differential equation but can be reduced to a system of two first-order equations. The exact solution to this problem is known, it is: $x(t) = 1 - \frac{\exp(\frac{t}{\epsilon})-1}{\exp(\frac{1}{\epsilon})-1}$. The problem is a good test case because the solution contains a boundary layer and we can test the effectiveness of the wavelet basis at capturing variations across different time scales. Note that as $\epsilon \to 0$ the gradient in the boundary layer becomes steeper. We compare the exact solution to two approximate solutions; one computed using Haar wavelets and the other using second-order finite differences. Approximate solutions are computed as well as relative errors for $\epsilon = 0.1$ and $\epsilon = 0.01$ and for various step sizes. The results are pictured in Figure 1. The wavelet method does indeed provide more accurate solutions for this problem.

3.2 Daubechies Wavelets

The more natural method for approximating differential equations is to differentiate the expansion series term by term and solve for the expansion coefficients, as discussed in Section 2.3. The difficulty here is determining the effect of differentiating the wavelet or scaling functions. The computations are not as straight forward as those for the Fourier basis, but there are successful schemes using differentiation.

In particular, Beylkin has constructed differentiation matrices for the Daubechies wavelets. Jameson outlines this construction [8] and constructs wavelet functions defined at boundaries so the method can be used to solve differential equations with non-periodic boundary conditions [9]. Their work utilizes scaling functions as the expansion terms, thus the term to plug into the differential equation would be $\mathbf{X} = \mathcal{V}^T\mathbf{V}$. Once again, the efficiency of the DWT makes it easy to transform back and forth between a function and its scaling coefficients. Furthermore, Jameson shows that the structure of the differentiation matrix is relatively simple and related to finite difference schemes of high order of accuracy [8]. Using $D(2)$ wavelets is identical to the second-order centered finite difference scheme discussed above. In general a $D(2n)$ approximates the centered difference scheme of order $2n$. This connection between Daubechies wavelets and centered differences is perhaps unsurprising because the Daubechies wavelets can be interpreted as a cascade of a difference filter and a low pass filter [12].

3.3 Improvements to Basic Schemes

The methods presented above give an overview of some basic techniques used to implement wavelet approximation of differential equations. As with all numerical methods, many refinements and improvements are constantly being introduced. For example, Goedecker and Ivanov introduce a family of wavelets called "interpolation wavelets." These wavelets are relatively easy to integrate and produce matrices with convenient structure for solving matrix equations [5]. Beylkin suggests implementing diagonal preconditioners to simplify the structure of the matrix equations that must be solved, thereby increasing the efficiency of the method [1].

Another frequently used technique is to choose the location of points on the computational grid to suit the solution. In the boundary layer problem analyzed in Section 3.1, it may be useful to concentrate grid points on the right end of the interval (within the boundary layer) instead of using a uniform increment in time. Cai and Wang introduce a method whereby grid points can be adaptively determined based on the problem [2].

In Figure 2 we show their results when solving the Inviscid Burger's Equation which is defined by the partial differential equation $u_x + uu_t = 0$. This is a well known PDE that models shock formation. By adaptively choosing their grid points, and by using wavelet methods that can localize the shock formation in space and time, they are able to produce remarkably smooth approximations to the solution even near the onset of the shock. Contrast this to results using a standard Fourier series expansion method (Figure 3). The Fourier modes have difficulty resolving the steep gradient near shock formation, and as a result spurious oscillations are visible in the approximation.

4 Application to Turbulent Flows

The methods discussed above show that wavelets are well-suited to approximating solutions with features that are localized in space and/or time such as shocks and boundary layers. This result is expected because the construction of wavelets enables them to decorrelate variations across time and scale whereas Fourier series methods do not share this localization property. One application that seems particularly well-suited to analysis by wavelet methods is the numerical simulation of turbulence. Turbulent flows are characterized by nonlinear effects that include the generation of variations on progressively smaller scales as well as shock formation. Fourier analysis is suited neither to solving the nonlinearity in the governing PDE nor to capturing the evolution of features across differing and localized scales.

Farge and Schneider [4] therefore turn to wavelet methods to solve the nonlinear Navier Stokes equations and to analyze the features of turbulent flows. Their wavelet of choice is the Coifman 12 Wavelet. The authors note that this wavelet is compactly supported and quasi-symmetric. The choice is based on their own research experience but no other justifications are given, so perhaps future work could investigate the effects of choosing a different wavelet.

To explain their choice to use wavelet methods they note the advantages we have discussed above, namely the efficiency of the DWT and the convenience of being able to solve sparse matrix systems in the wavelet domain. They also note that wavelets can be used to define a local (in space) energy spectrum that can be used to estimate which local features contribute the most energy to the global energy spectrum. This can be useful in separating underlying patterns in the flow from the random turbulent motion. Once again Fourier analysis cannot be used to define such a measure because the energy spectrum in the Fourier domain would automatically be spread across all of physical space.

Lastly, Farge and Schneider note that investigations of turbulence frequently attempt to isolate so-called "coherent structures" in the flow. Their analysis is focused on vortices in the flow. These structures are important in the mixing and transport of the fluid but can be difficult to detect and study because turbulent flows are characterized by seemingly random fluid motions that appear as noise in the signal. Fortunately wavelets are an effective tool for denoising signals [12]. Farge and Schneider assume the "incoherent" portions of the flow can be modeled as uncorrelated Gaussian white noise and use thresholding methods developed by Donoho and Johnstone to denoise the wavelet coefficients that they have computed in solving the governing PDE. The results are pictured in the attached figures. Figure 4 displays the total computed vorticity field. The signal is then denoised so that Figure 5 displays only the estimated coherent structures. In Figure 6 the underlying incoherent flow is pictured. The reader can see that the vortex tubes and other coherent structures are visible in Figure 5, whereas Figure 6 appears more homogeneous and with few features that would indicate the presence of non-random dynamics.

5 Conclusion

This paper has provided an overview of the use of wavelets in computing approximate solutions for differential equations. We have seen that the familiar characteristics of wavelets (the computational efficiency of the DWT and the localization of wavelets in time or space) make them a desirable orthogonal basis with which to expand functions. In particular we have seen that wavelet transforms can outperform traditional finite difference and Fourier spectral methods in approximating solutions to differential equations that feature localized variations such as boundary layers and shocks.

We have also discussed recent work that applies wavelet methods to turbulent flows. Here wavelets have been used to solve the partial differential equation that describes turbulent flows. The computed wavelet coefficients can then be denoised using standard techniques so that further insight into the complex structure of turbulence can be attained.

There are difficulties associated with wavelet methods. In particular, it is more difficult to approximate the effect of differential and integral operators on wavelet and scaling functions than Fourier basis functions. But based on results presented above it seems that there are many applications for which the characteristics of wavelets recommend them to numerical solutions of differential equations.

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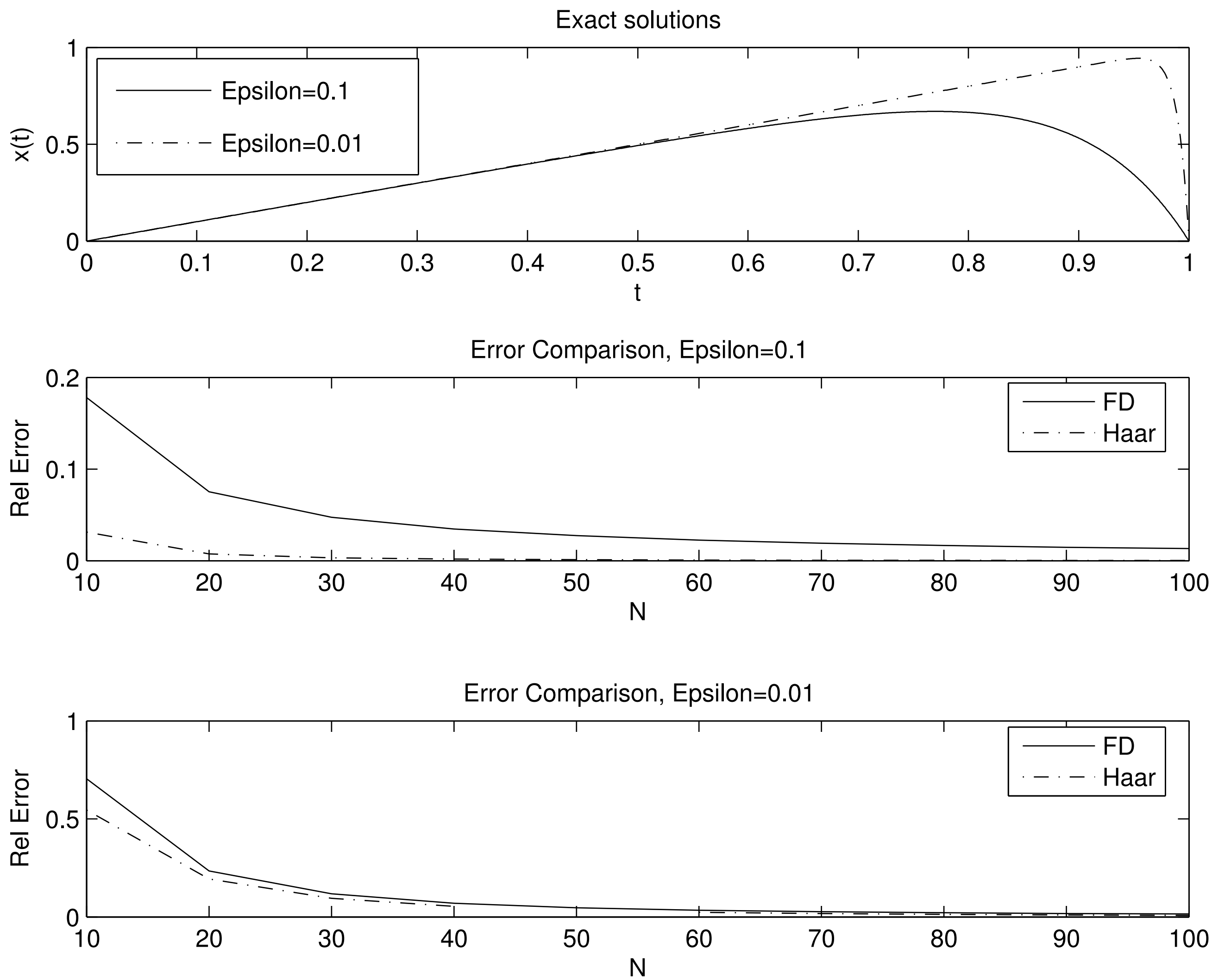

Figure 1: Approximate Solutions of Differential Equation with Boundary Layer using Haar Wavelets and Finite Differences. N is number of grid points and Rel Error is error in Two-Norm of approximate solutions compared to exact solution.

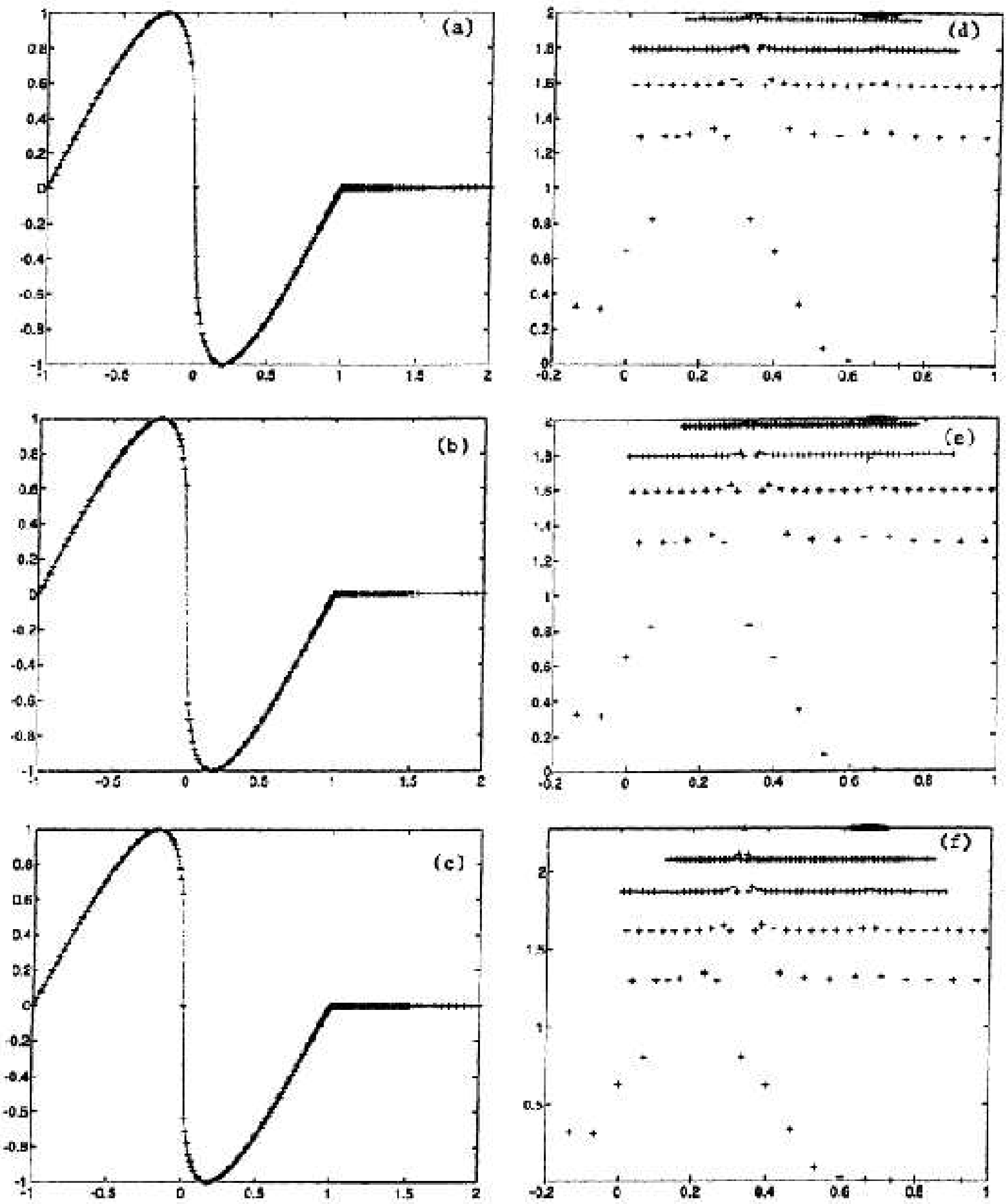

FIG. 14. *Adaptive wavelet collocation solution of inviscid Burger's equation* (6.5) *with initial condition* (6.6) *at* $t = 0.3, 0.318, 0.319$ *with* $L = 15$, $J = 8$, *and error tolerance* $\epsilon = 10^{-4}$. *The number of collocation points* $N = 292, 295$, *and* 303 *at time* $t = 0.3, 0.318$, *and* 0.319, *respectively.* (a)–(c) *plus sign—numerical solutions;* (d)–(f) *wavelet coefficients at all levels.*

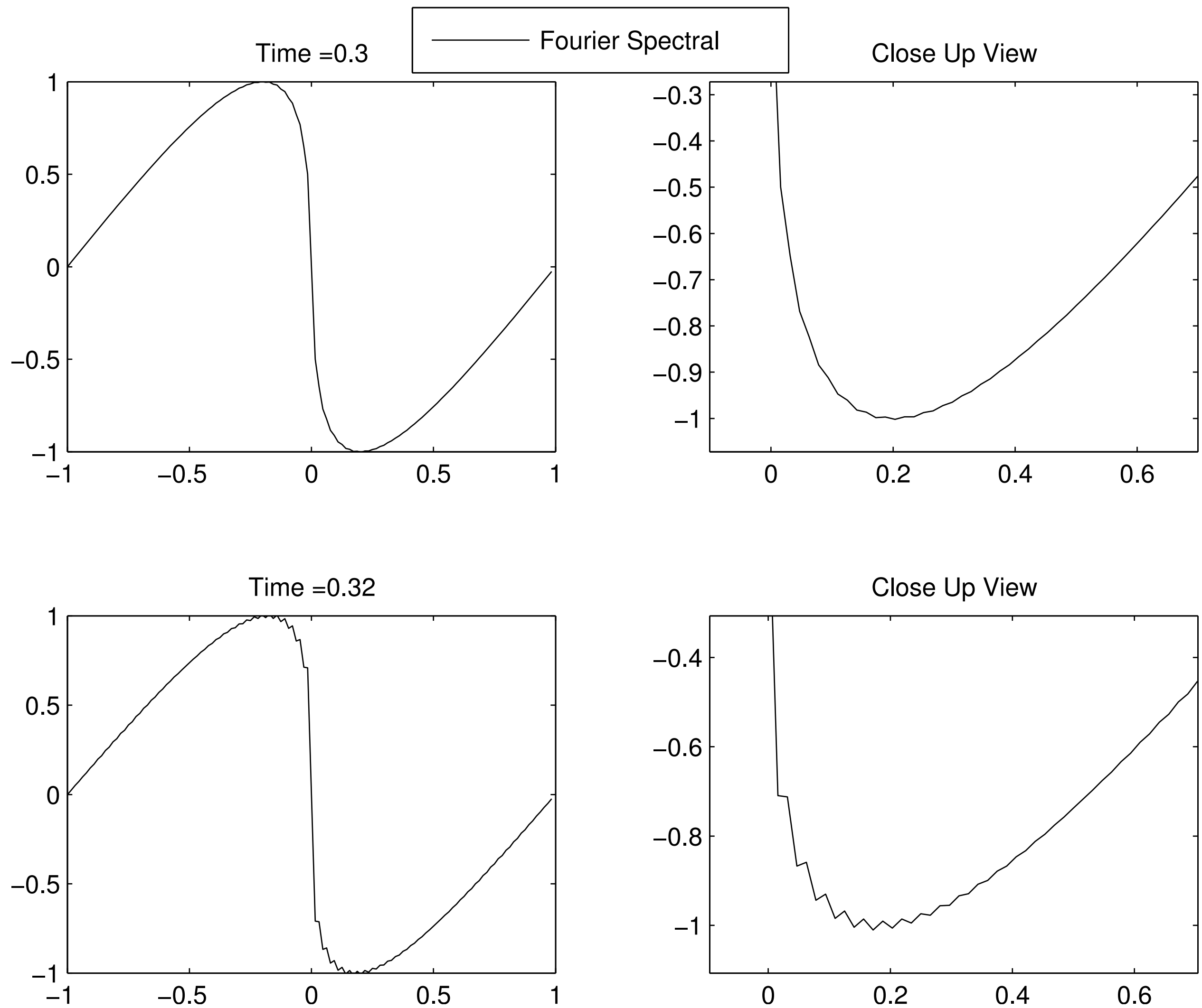

Figure 3: Approximate Solution of Burger's Equation using Fourier Spectral Method. Solution is for 128 grid points, an attempt using 256 (Cai and Wang use 300) was also tried but the solution was unstable.

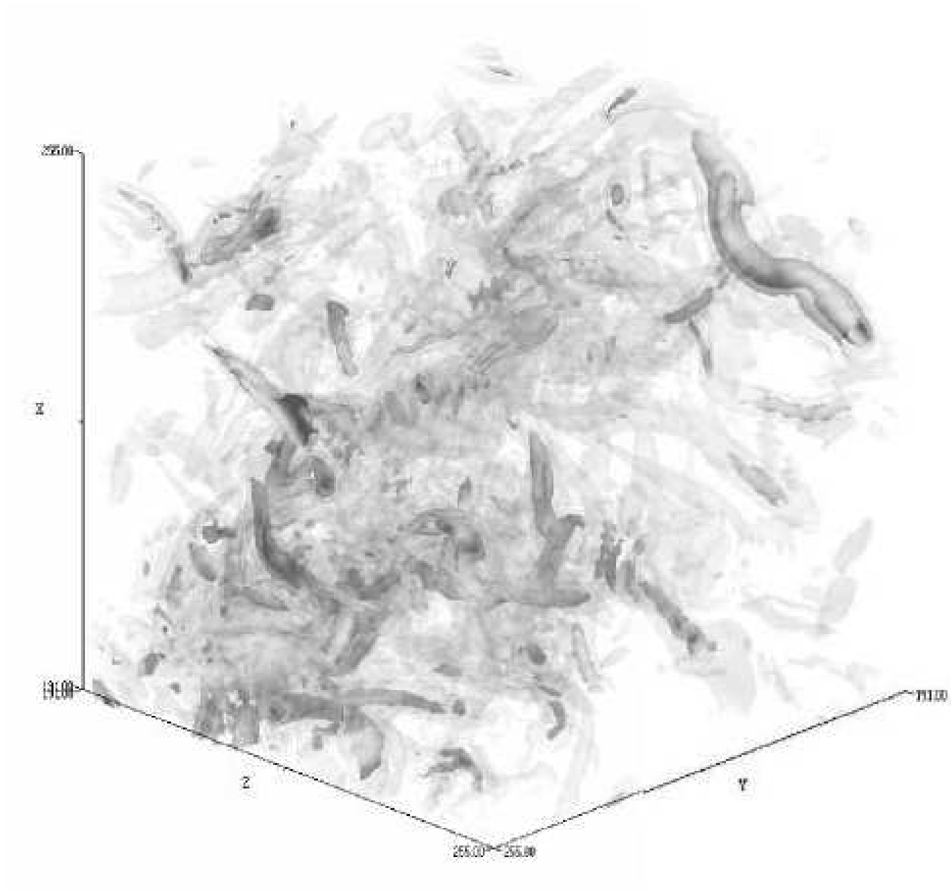

Figure 4: Total Vorticity Field in Turbulent Flow. Figure reproduced from [4]

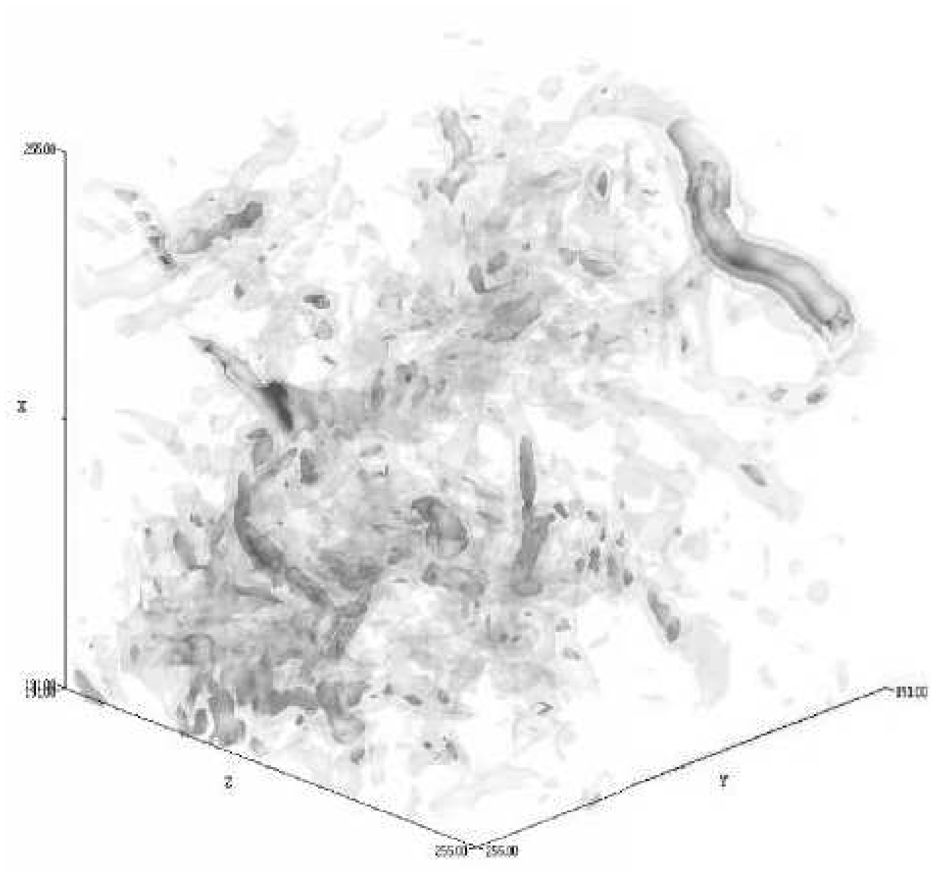

Figure 5: Coherent Vorticity in Turbulent Flow. Figure reproduced from [4]

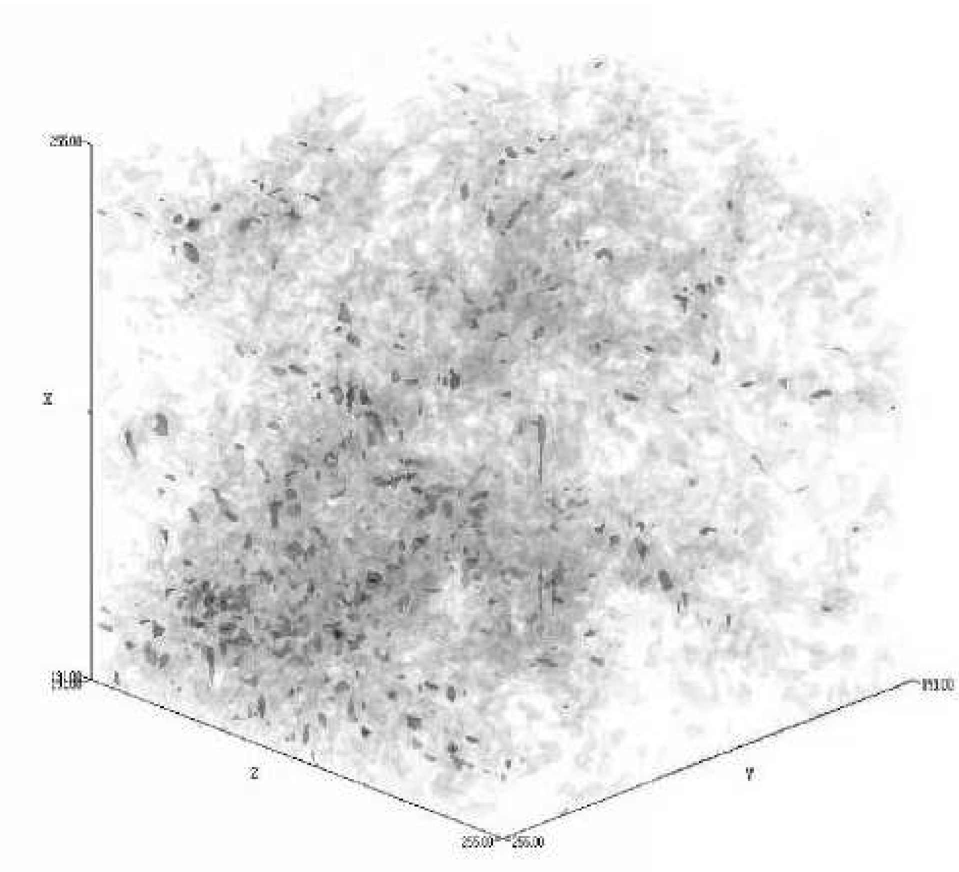

Figure 6: Incoherent Vorticity in Turbulent Flow. Figure reproduced from [4]