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A NOTE ON ISOCHORIC PROBLEMS IN COMPRESSIBLE FINITE ELASTICITY

Eleftherios Kirkinis

University of Glasgow
Department of Mathematics
Glasgow G12 8QW, UK
e-mail:ek@maths.gla.ac.uk

Ray W. Ogden

University of Glasgow
Department of Mathematics
Glasgow G12 8QW, UK
e-mail:rwo@maths.gla.ac.uk

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Abstract. *For compressible isotropic elastic solids we consider certain classes of isochoric deformations that represent simple shear locally. The maximal class of strain-energy functions for which all these deformations are possible is then characterized and illustrated with specific examples.*

1 INTRODUCTION

Considerable activity has been devoted recently to the study of isochoric problems in the nonlinear theory of compressible elasticity, mainly for isotropic materials. A review of this work has been provided by Horgan.[1]

With this background in mind, it was noted by Kirkinis and Ogden[2] that the compressible version of the Varga material, introduced by Haughton[3] and Carroll,[4] namely

$$\tilde{W} = a(i_1 - 3) + b(i_2 - 3) + h(i_3), \tag{1}$$

has the interesting property that it admits a number of deformations, including pure torsion,[5] pure azimuthal shear[6] and pure axial shear[7] that, locally, are simple shear deformations. In (1), a and b are constants, i_1, i_2, i_3 are the principal invariants of the stretch tensor $\mathbf{U}$ (or $\mathbf{V}$) arising in the polar decomposition $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ of the deformation gradient tensor $\mathbf{F}$, and h is a function of i_3. It was also shown by Kirkinis and Ogden[8] that, with $a = b$, (1) admits the more complex deformation of helical shear, which is also locally a simple shear deformation.

In Section 2 of the present article, with reference to cylindrical polar coordinates, we examine a number of isochoric deformations that locally have the character of simple shear. These are (i) helical shear (which includes pure azimuthal shear and pure axial shear as special cases), (ii) anti-plane shear (which includes pure axial shear as a special case), and (iii) generalized azimuthal shear (which includes pure azimuthal shear and pure torsion as special cases).

In Section 3, we then show that the two necessary and sufficient conditions given by Kirkinis and Ogden[8] that the strain-energy function must satisfy for it to admit helical shear also ensure that it admits the other deformations listed above. The conditions are expressed in terms of the derivatives of the strain-energy function with respect to the invariants I_1, I_2, I_3 of the Cauchy-Green deformation tensor $\mathbf{B} = \mathbf{FF}^T$, and a specific example of a strain-energy function satisfying these conditions is given.

Alternative expressions for the necessary and sufficient conditions, in terms of the invariants i_1, i_2, i_3, are then given in Section 4 and a further class of energy functions is exemplified. This includes (1), with $b = a$, as a particular member.

2 LOCALLY SIMPLE SHEAR DEFORMATIONS

First we note that the principal invariants of the Cauchy-Green deformation tensor $\mathbf{B}$ are defined by

$$I_1 = \mathrm{tr}\,(\mathbf{B}), \quad I_2 = I_3 \mathrm{tr}\,(\mathbf{B}^{-1}), \quad I_3 = \det(\mathbf{B}). \tag{2}$$

In terms of the principal stretches $\lambda_1, \lambda_2, \lambda_3$ these may be rewritten as

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{3}$$

A characterization of simple shear based on the use of these invariants was given by Truesdell and Toupin[9] (p. 295), as follows: *A homogeneous isochoric strain* ($I_3 = 1$) *may be regarded as simple shear followed by or preceded by a rotation about the axis of shear if and only if*

$$I_1 = I_2; \tag{4}$$

equivalently, if and only if it is plane. The amount of shear is $\gamma = \sqrt{I_1 - 3}$. However, we point out here that equation (4) is also characteristic of a *pure shear* deformation and hence equation (4), with $I_3 = 1$, does not distinguish between simple shear and pure shear.

For non-homogeneous deformations that reduce locally to simple shear Truesdell and Toupin[9] (p. 296) gave the characterization: *At a point where* $I_3 = 1$ and $I_1 = I_2$, *the measures of strain and rotation have the same values as for a simple shear of amount* $\gamma = \sqrt{I_1 - 3}$, *in the plane normal to the principal direction whose stretch is* 1. This characterization applies to all the deformations studied in the present paper.

At this point we recall that a homogeneous deformation with deformation gradient of the form

$$\mathbf{F} = \mathbf{I} + \gamma \mathbf{m} \otimes \mathbf{n}, \tag{5}$$

where $\mathbf{I}$ is the identity tensor and $\mathbf{m}$ and $\mathbf{n}$ are mutually perpendicular unit vectors, is called a *simple shear*, γ being the *amount of shear* and $\mathbf{m}$ the *direction of shear*. The vectors $\mathbf{m}$ and $\mathbf{n}$ define the *plane of shear*, and planes normal to $\mathbf{n}$ are called *glide planes*. The principal stretches associated with (5) are expressible in the form

$$\lambda_1 = \lambda, \quad \lambda_2 = \lambda^{-1}, \quad \lambda_3 = 1, \quad \lambda - \lambda^{-1} = \gamma \tag{6}$$

provided we take $\gamma \geq 0$ to correspond to $\lambda \geq 1$.

It is well known[10,11] that two simple shears combine to form a third simple shear if and only if each direction of shear is parallel to both families of glide planes. This result also applies to locally simple shear deformations, and we make use of it here. For the purposes of this article, $\mathbf{m}$ and $\mathbf{n}$ will be drawn from the set $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ of cylindrical polar basis vectors and we consider the following three combinations of (local) simple shears.

(i) **Helical shear**. Helical shear has been examined by Beatty and Jiang[12] and Kirkinis and Ogden[8] for isotropic materials and by Jiang and Beatty[13] in respect of transversely isotropic materials. It has the form

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z + w(R), \tag{7}$$

where r, θ, z and R, Θ, Z are cylindrical polar coordinates of a material point in the current and reference configurations respectively, and g and w are functions of R. The deformation gradient has the form

$$\mathbf{F} = \mathbf{I} + \gamma_\theta \, \mathbf{e}_\theta \otimes \mathbf{E}_R + \gamma_z \, \mathbf{e}_z \otimes \mathbf{E}_R, \tag{8}$$

where

$$\gamma_\theta = R\dot{g}, \quad \gamma_z = \dot{w}, \tag{9}$$

and a superposed dot indicates differentiation with respect to R. Following Ogden[10] we may decompose equation (8) in the form

$$\mathbf{F} = (\mathbf{I} + \gamma_\theta\, \mathbf{e}_\theta \otimes \mathbf{e}_r)(\mathbf{I} + \gamma_z\, \mathbf{e}_z \otimes \mathbf{e}_r)\mathbf{Q}, \tag{10}$$

where $\mathbf{Q}$ denotes the rotation tensor

$$\mathbf{Q} = \mathbf{e}_r \otimes \mathbf{E}_R + \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \mathbf{e}_z \otimes \mathbf{E}_Z. \tag{11}$$

It is evident that this deformation reduces locally to simple shear with direction of shear along the vector $\gamma_\theta\mathbf{e}_\theta + \gamma_z\mathbf{e}_z$, amount of shear $\gamma = (\gamma_\theta^2 + \gamma_z^2)^{1/2}$, and glide planes normal to $\mathbf{e}_r$, as was shown in.[11]

(ii) **Anti-plane shear**. In cylindrical coordinates this deformation has the form

$$r = R, \quad \theta = \Theta, \quad z = Z + u(R, \Theta), \tag{12}$$

where u is the axial displacement function. This was studied by Knowles[14] (in Cartesian coordinate form) and Agarwal[15] for isotropic materials and by Tsai and Fan[16] for transversely isotropic materials. The deformation gradient has the form

$$\mathbf{F} = \mathbf{I} + \mathbf{e}_z \otimes (\gamma_r\mathbf{E}_R + \gamma_\theta\mathbf{E}_\Theta), \tag{13}$$

where

$$\gamma_r = u_R, \quad \gamma_\theta = u_\Theta/R, \tag{14}$$

and the subscripts R and Θ denote partial derivatives. Note that γ_θ in (14) differs from that in (9). In decomposed form (14) reads

$$\mathbf{F} = (\mathbf{I} + \gamma_r\, \mathbf{e}_z \otimes \mathbf{e}_r)(\mathbf{I} + \gamma_\theta\, \mathbf{e}_z \otimes \mathbf{e}_\theta)\mathbf{Q}, \tag{15}$$

where $\mathbf{Q}$ is given by (11). The two axial shears correspond to local simple shears of amounts γ_r and γ_θ in planes normal to $\mathbf{e}_\theta$ and $\mathbf{e}_r$ respectively. The resultant deformation consists of the rotation $\mathbf{Q}$ and a local simple shear of amount $\gamma = (\gamma_r^2 + \gamma_\theta^2)^{1/2}$ in the direction $\mathbf{e}_z$ with $\gamma_r\mathbf{e}_r + \gamma_\theta\mathbf{e}_\theta$ normal to the glide planes.

(iii) **Generalized azimuthal shear**. This deformation is described by

$$r = R, \quad \theta = \Theta + g(R, Z), \quad z = Z, \tag{16}$$

and includes as special cases pure torsion, with $g = \tau Z$ and τ constant, and pure azimuthal shear, for which $g = g(R)$. The deformation gradient has the form

$$\mathbf{F} = \mathbf{I} + \mathbf{e}_\theta \otimes (\gamma_r \mathbf{E}_R + \gamma_z \mathbf{E}_Z), \tag{17}$$

where

$$\gamma_r = R g_R, \quad \gamma_z = R g_Z, \tag{18}$$

the subscripts R and Z denote partial derivatives, and we note that γ_r and γ_z are not the same as in (14) and (9) respectively. Equation (17) may be decomposed as

$$\mathbf{F} = (\mathbf{I} + \gamma_r \, \mathbf{e}_\theta \otimes \mathbf{e}_r)(\mathbf{I} + \gamma_z \, \mathbf{e}_\theta \otimes \mathbf{e}_z)\mathbf{Q}, \tag{19}$$

where again $\mathbf{Q}$ is given by (11). The azimuthal and torsional shears correspond to local simple shears of amounts γ_r and γ_z in planes normal to $\mathbf{e}_z$ and $\mathbf{e}_r$ respectively. The resultant deformation consists of the rotation $\mathbf{Q}$ and a local simple shear of amount $\gamma = (\gamma_r^2 + \gamma_z^2)^{1/2}$ in the direction $\mathbf{e}_\theta$, with $\gamma_r \mathbf{e}_r + \gamma_z \mathbf{e}_z$ normal to the glide planes.

In each of (i)–(iii) above the composite deformation is a local simple shear of amount equal to the square root of the sum of the squares of the constituent local simple shears. For the case of homogeneous strains this was first observed by Love[17] (p. 50).

We conclude this section with a description of tests required to characterize the composition of two simple shears: when two simple shears combine to give a new deformation, then the following are equivalent:

(a) The composite deformation is a simple shear.

(b) The deformation gradients of the constituent simple shears commute.

(c) Each direction of shear is parallel to both families of glide planes.

(d) For the composite shear, $I_1 = I_2$ and $I_3 = 1$.

The equivalence of (a), (b) and (c) was established by Ogden *et al.*[11]; see also Ogden.[10] The equivalence of (a) and (d) follows from the theorem of Truesdell and Toupin, which holds for any simple shear. Note that it is easy to show that no non-trivial pure shear can commute with a simple shear so that (d) does not imply that the composite deformation is a pure shear.

3 STRAIN ENERGIES ADMITTING LOCAL SIMPLE SHEAR DEFORMATIONS

In this section we examine the necessary and sufficient conditions on the strain-energy function that have been derived previously in the literature for the deformations discussed in Section 2 to be admitted. Specifically we show that strain-energy functions for which helical shear is admitted can also support the other deformations listed in Section 2.

We denote by $\bar{W}(I_1, I_2, I_3)$ the strain energy of a compressible isotropic elastic material regarded as a function of the three invariants (2).

Helical shear deformations. It was shown by Kirkinis and Ogden[8] that helical shear can be sustained if and only if the two conditions

$$\bar{W}_1 = \bar{W}_2, \quad 4I\bar{W}_1 + 4\bar{W}_3 - \bar{W} = 0 \tag{20}$$

hold, where $I = I_1 = I_2 = 3 + \gamma^2, I_3 = 1$, γ is as given in Section 2(i) and the subscripts on $\bar{W}$ signify differentiation with respect to I_1, I_2, I_3. Previously, Beatty and Jiang[12] obtained the condition

$$\bar{W}_1 + 2I(\bar{W}_{11} + \bar{W}_{12}) + 2\bar{W}_{13} + 2\bar{W}_{23} = 0, \tag{21}$$

involving the second derivatives of $\bar{W}$, which was shown by Kirkinis and Ogden[8] to be equivalent to $(20)_2$.

Axisymmetric azimuthal shear and axial shear. Pure azimuthal shear was examined by Haughton,[6] Polignone and Horgan,[18] Beatty and Jiang[19] and Jiang and Ogden[20] amongst others, and pure axial (axisymmetric anti-plane) shear by Polignone and Horgan,[21] Jiang and Beatty[22] and Jiang and Ogden.[7] Since these are special cases of helical shear they are each admitted by the requirements (20).

Pure torsion. A necessary and sufficient condition for the strain-energy function to support pure torsion was obtained by Polignone and Horgan[5] in the form

$$\bar{W}_{11} + I\bar{W}_{12} + (I-1)\bar{W}_{22} + \bar{W}_{31} + \bar{W}_{32} + \bar{W}_2 - \frac{1}{2}\bar{W}_1 = 0, \tag{22}$$

evaluated at $I = I_1 = I_2 = 3 + \gamma^2, I_3 = 1$, where $\gamma = \tau R$. Pure torsion has also been examined recently by Kirkinis and Ogden[2] with the additional requirement that the traction on the lateral surface of the cylinder should vanish. This imposes more restrictive conditions on the strain-energy function than (22) and we do not consider these here. If the equality $(20)_1$ holds then it follows on differentiation with respect to I that $\bar{W}_{11} = \bar{W}_{22}$, and hence equation (22) reduces to (21) and therefore to $(20)_2$.

Non-axisymmetric anti-plane shear. Necessary and sufficient conditions on the strain-energy function to support the non-axisymmetric anti-plane shear (12) were obtained, using Cartesian coordinates, by Knowles[14] in the form

$$b\bar{W}_1 + (b-1)\bar{W}_2 = 0, \tag{23}$$

$$\bar{W}_{11} + I\bar{W}_{12} + (I-1)\bar{W}_{22} + \bar{W}_{13} + \bar{W}_{23} + \frac{1}{2}\bar{W}_2 = 0, \tag{24}$$

both evaluated for $I = I_1 = I_2 = 3 + \gamma^2, I_3 = 1$, where γ is the resultant simple shear identified in Section 2(ii) and b is some constant. On inspection the first of these conditions is identified with $\bar{W}_1 = \bar{W}_2$ by setting $b = 1/2$, while the second then reduces to (21) and therefore to $(20)_2$.

General azimuthal shear. For the deformation discussed in Section 2(iii) it can be shown, after a lengthy calculation, that on requiring the strain-energy function to satisfy the condition $\bar{W}_1 = \bar{W}_2$ the three equilibrium equations are satisfied if and only if the condition $(20)_2$ is met together with the equation

$$\frac{\partial}{\partial R}\left(R^2\gamma_r\bar{W}_1\right) + \frac{\partial}{\partial Z}\left(R^2\gamma_z\bar{W}_1\right) = 0. \tag{25}$$

Thus, for any strain-energy function satisfying the two conditions in (20) equation (25) provides an equation for the determination of $g(R, Z)$. Solutions of this equation and associated boundary-value problems will be examined in detail elsewhere.

We now state the main result for this article: *any strain-energy function that satisfies the conditions* (20) *that are necessary and sufficient for it to admit the helical shear deformation* (7) *is also able to support the other local simple shear deformations discussed in* Section 2.

3.1 An illustrative energy function

Kirkinis and Ogden[8] derived several classes of strain-energy functions for which the necessary and sufficient conditions (20) for the helical shear deformation to be supported are met. One such example is

$$\bar{W}(I_1, I_2, I_3) = \frac{3\mu}{4k3^k}\left[I_1^k h_1(I_3) + I_2^k h_2(I_3)\right] + h_3(I_3), \tag{26}$$

where $k \neq 0$ is a disposable parameter, the functions h_1, h_2 and h_3 satisfy

$$h_1(1) = h_2(1) = 1, \quad h_1'(1) + h_2'(1) = 1/2 - k, \tag{27}$$

$$h_3(1) = -\frac{3\mu}{2k}, \quad h_3'(1) = -\frac{3\mu}{8k}, \tag{28}$$

and $\mu\,(>0)$ is the shear modulus of the material in the natural configuration. This energy function, and the others[8] derived by Kirkinis and Ogden, also admit the anti-plane shear and generalized azimuthal shear deformations discussed in Section 2. For the special case $k=1$ it is worth noting that (26) corresponds to a generalized version of the Hadamard material, which was used by Agarwal,[15] Jiang and Beatty[22] and Polignone and Horgan[5,18,21] amongst others. We recall that the Hadamard material corresponds to (26) with $h_1(I_3)$ and $h_2(I_3)$ set as constants.

4 ENERGY FUNCTIONS IN TERMS OF THE STRETCH INVARIANTS

The necessary and sufficient conditions (20) for helical shear can be written in terms of the invariants of the stretch tensors $\mathbf{U}, \mathbf{V}$, which, in terms of the stretches, are given by

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{29}$$

Thus, following Kirkinis and Ogden,[8] we see that (20) are equivalent to

$$\tilde{W}_1 = \tilde{W}_2, \quad 2i\tilde{W}_1 + 2\tilde{W}_3 - \tilde{W} = 0, \tag{30}$$

evaluated for $i = i_1 = i_2 = 1 + \sqrt{4+\gamma^2}, i_3 = 1$. On differentiation of $(30)_2$ with respect to i and use of $(30)_1$ we obtain

$$i(\tilde{W}_{11} + \tilde{W}_{12}) + \tilde{W}_{13} + \tilde{W}_{23} = 0, \tag{31}$$

which is equivalent to $(30)_2$. It is easy to see that by requiring $\tilde{W}_1 = \tilde{W}_2$, the corresponding necessary and sufficient conditions on the strain energy for azimuthal shear,[20] axial shear[7] and pure torsion[2] reduce to $(30)_2$.

In Kirkinis and Ogden[8] we derived a class of strain-energy functions that satisfy the requirements (30). This has the form

$$\tilde{W}(i_1, i_2, i_3) = \frac{3\mu}{3^k k}\left[i_1^k h_1(i_3) + i_2^k h_2(i_3)\right] + h_3(i_3), \tag{32}$$

where $k \neq 0$,

$$h_1(1) = h_2(1) = 1, \quad h_1'(1) + h_2'(1) = 1 - k, \tag{33}$$

and

$$h_3(1) = -\frac{6\mu}{k}, \quad h_3'(1) = -\frac{3\mu}{k}. \tag{34}$$

For the special case $h_1'(i) = h_2'(i) \equiv 0$, so that $k=1$, equation (32) reduces to the Varga material, which was discussed in the Introduction, but with $a = b = \mu$. As noted by Kirkinis and Ogden,[8] the energy functions (32) and (26) lead to different solutions of the equilibrium equations.

5 CONCLUDING REMARKS

In the preceding sections we have examined necessary and sufficient conditions on the form of compressible isotropic elastic strain-energy functions required for them to support each of several isochoric deformations that locally have the form of simple shears. Strain-energy functions satisfying these conditions admit all the considered deformations and therefore, for this class of materials, the deformations can be considered as controllable. However, it should be emphasized that the fact that a strain-energy admits such a deformation does not in general guarantee that the deformation actually exists for the considered material. Establishment of existence requires further analysis, as exemplified in Jiang and Ogden[7,20] and Kirkinis and Ogden.[8]

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