Functions

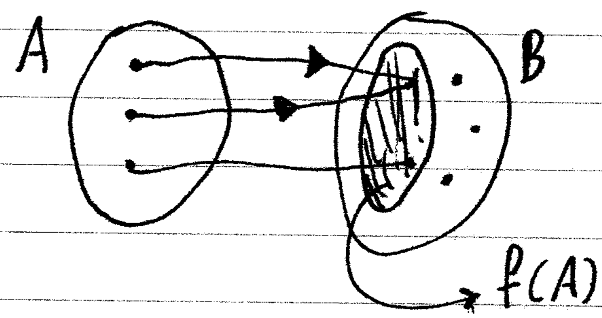
A function $f: A \to B$ is a rule that assigns each element $x \in A$ to one and only one element $f(x) \in B$.

$A \to$ domain of function f

$B \to$ destination set

The range $f(A)$ is the set

$$f(A) = \{f(x) \mid x \in A\}.$$



To define a function we must give

1) The domain A
2) The rule $f(x)$.

If the domain is not given, then by default we understand that the widest possible subset of $\mathbb{R}$ is being used. The notation $f // A$ means that A is the domain of f.

e.g. $f(x) = (1/x) // \mathbb{R} - \{0\}$ defines f with rule $1/x$ and domain $\mathbb{R} - \{0\}$.

Consequently, if f is

1) Polynomial : $f(x) = a_n x^n + \dots + a_1 x + a_0$
   we take $A = \mathbb{R}$.
2) Rational : $f(x) = P(x)/Q(x)$
   with P, Q polynomials
   we take $A = \mathbb{R} - \{x \in \mathbb{R} \mid Q(x) = 0\}$
3) Irrational : $f(x) = \sqrt{g(x)}$
   we take $A_f = \{x \in \mathbb{R} \mid g(x) \geq 0\} \cap A_g$

Algebra of functions

Let $f_1 : A_1 \to \mathbb{R}$ and $f_2 : A_2 \to \mathbb{R}$.

- Function equality

$$f_1 = f_2 \iff \begin{cases} A_1 = A_2 = A \\ \forall x \in A : f_1(x) = f_2(x) \end{cases}$$

- Function sum

$$f = f_1 + f_2 \iff \begin{cases} A = A_1 \cap A_2 \\ \forall x \in A : f(x) = f_1(x) + f_2(x) \end{cases}$$

- Function product

$$f = f_1 f_2 \iff \begin{cases} A = A_1 \cap A_2 \\ \forall x \in A : f(x) = f_1(x) f_2(x) \end{cases}$$

Function quotient

$$f = f_1/f_2 \iff \begin{cases} A = A_1 \cap \left[A_2 - \{ x \in \mathbb{R} \mid f_2(x) \} \right] \\ f(x) = f_1(x)/f_2(x), \ \forall x \in A \end{cases}$$

Scalar product

$$f = a f_1, \ a \in \mathbb{R} \iff \begin{cases} A = A_1 \\ f(x) = a f(x), \ \forall x \in A \end{cases}$$

Negation

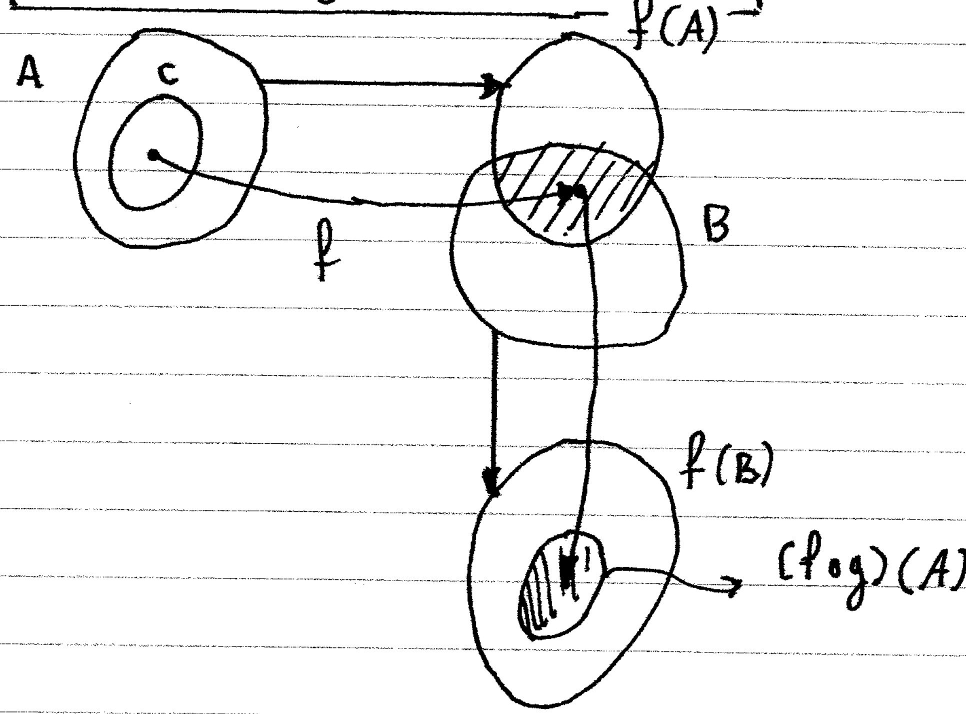
$$f = -f_1 \iff \begin{cases} A = A_1 \\ f(x) = -f_1(x), \ \forall x \in A \end{cases}$$

Subtraction : $f = f_1 - f_2 = f_1 + (-f_2)$.

Composition of functions.

Let $f /\!/ A$ and $g /\!/ B$ be two functions and assume that $f(A) \cap B \neq \emptyset$. Then we can define a new function $fog /\!/ C$ as follows:

$$\boxed{\begin{array}{l} C = \{x \in \mathbb{R} \mid x \in A \wedge f(x) \in B\} \\ (fog)(x) = f(g(x)) \end{array}}$$



Method: To find fog we

(1) Solve the system of equations $\begin{cases} x \in A \\ f(x) \in B \end{cases}$ to find C

(2) Calculate the rule $(fog)(x) = f(g(x))$.

Inverse function

Let $f: A \to \mathbb{R}$. A function $g: f(A) \to \mathbb{R}$ is the inverse of f, if and only if ~~$(f \circ g)(x) = x, \forall$~~ $(g \circ f)(x) = x$, $\forall x \in A$. Then we write $g = f^{-1}$.

$$\boxed{g = f^{-1} \Leftrightarrow \forall x \in A : g(f(x)) = x}$$

<u>Theorem</u> : A function f has an inverse f^{-1}, if and only if it satisfies the condition.

$$\boxed{\forall x_1, x_2 \in A : \left(f(x_1) = f(x_2) \Rightarrow x_1 = x_2\right)}$$

<u>Theorem</u> : If $g = f^{-1} \Rightarrow \forall x \in f(A) : f(g(x)) = x$

<u>Def</u> : A function f is strictly increasing if and only if $\forall x_1, x_2 \in A : (x_1 < x_2 \Rightarrow f(x_1) < f(x_2))$
A function f is strictly decreasing if and only if $\forall x_1, x_2 \in A : (x_1 < x_2 \Rightarrow f(x_1) > f(x_2))$.
In either case we say that f is strictly <u>monotone.</u>

Theorem

a) If f strictly increasing in A $\Rightarrow \begin{cases} f \text{ has an inverse } f^{-1} \\ f^{-1} \text{ strictly increasing} \end{cases}$

b) If f strictly decreasing in A $\Rightarrow \begin{cases} f \text{ has an inverse } f^{-1} \\ f^{-1} \text{ strictly decreasing.} \end{cases}$