

Vector Spaces

Let V be a set where we define

1) Addition

$$\forall u, v \in V \longrightarrow u+v \in V$$

2) Scalar Multiplication.

$$\forall a \in \mathbb{R} : \forall u \in V \longrightarrow au \in V$$

Def: $(V, +, \cdot)$ is a vector space if it satisfies:

1) Closure

$$\forall u, v \in V : u+v \in V$$

$$\forall a \in \mathbb{R} : \forall u \in V : au \in V$$

2) $(V, +)$ abelian group

$$\forall u, v, w \in V : (u+v)+w = u+(v+w)$$

$$\forall u, v \in V : u+v = v+u$$

$$\exists \mathbf{0} \in V : \mathbf{0}+u = u+\mathbf{0} = u, \forall u \in V$$

$$\forall u \in V : \exists v \in V : u+v = \mathbf{0}$$

3) The scalar multiplication satisfies

$$\forall a \in \mathbb{R} : \forall u, v \in V : \lambda(u+v) = \lambda u + \lambda v$$

$$\forall a, b \in \mathbb{R} : \forall u \in V : (a+b)u = au + bu$$

$$\forall a, b \in \mathbb{R} : \forall u \in V : a(bu) = (ab)u$$

$$\forall u \in V : 1u = u$$

Def: Let $(V, +, \cdot)$ be a vector space.

a) An inner product is a mapping

$$\forall u, v \in V \rightarrow (u, v) \in \mathbb{R}$$

such that

$$\forall u, v \in V: (u, v) = (v, u)$$

$$\forall u, v, w \in V: (u, v+w) = (u, v) + (u, w)$$

$$\forall a \in \mathbb{R}: \forall u, v \in V: a(u, v) = (au, v)$$

$$\forall u \in V - \{0\}: (u, u) > 0$$

b) A norm is a mapping

$$\forall u \in V \rightarrow \|u\| \in \mathbb{R}_0^+$$

such that

$$u = 0 \Rightarrow \|u\| = 0$$

$$u \neq 0 \Rightarrow \|u\| > 0$$

$$\forall a \in \mathbb{R}: \forall u \in V: \|au\| = |a| \cdot \|u\|$$

$$\forall u, v \in V: \|u+v\| \leq \|u\| + \|v\|$$

Thm: If (u, v) is an inner product

a) $\forall u, v \in V: |(u, v)|^2 \leq (u, u) \cdot (v, v)$

b) $\|u\| = (u, u)$ defines a norm.

Linear independence and orthogonality.

The

Def: The vectors $\{v_1, v_2, \dots, v_k\}$ are linearly independent iff

$$a_1 v_1 + a_2 v_2 + \dots + a_k v_k \Rightarrow a_1 = a_2 = \dots = a_k = 0$$

Def: The set $B = \{v_1, v_2, \dots, v_k\} \subset V$ is a basis of V iff.

a) v_1, \dots, v_k linearly independent

b) $V = \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n \mid a_1, a_2, \dots, a_n \in \mathbb{R}\}$

↳ All basis of a vector space V have the same number of elements, and we call that number n the dimension of V : $n = \dim V$.

Def: a) Two vectors $u, v \in V$ are orthogonal iff:
 $u \perp v \Leftrightarrow (u, v) = 0$.

b) The vectors $\{u_1, u_2, \dots, u_n\}$ are orthogonal iff:
 $a \neq b \Rightarrow u_a \perp u_b, \forall a, b \in \{1, 2, \dots, n\}$

c) The vectors $\{u_1, u_2, \dots, u_n\}$ are orthonormal iff $\{u_1, u_2, \dots, u_n\}$ orthogonal and
 $(u_k, u_k) = 1, \forall k \in \{1, 2, \dots, n\}$

↗ If $\{u_1, u_2, \dots, u_n\}$ orthogonal \Rightarrow
 $\Rightarrow \{u_1, u_2, \dots, u_n\}$ linearly independent.

If $\{u_1, u_2, \dots, u_n\}$ orthogonal $\Rightarrow \{u_1, u_2, \dots, u_n\}$
d. $\dim V = n$ basis of V .

↗ If $\{u_1, u_2, \dots, u_n\}$ orthogonal basis of V

then

$$\forall v \in V: v = \sum_{k=1}^n \frac{(v, u_k)}{(u_k, u_k)} u_k$$

From a linearly independent basis $\{u_1, u_2, \dots, u_n\}$
one may construct an orthogonal basis
 $\{v_1, v_2, \dots, v_n\}$ by the Gram-Schmidt iteration:

$$v_1 = u_1$$
$$v_{k+1} = u_{k+1} - \sum_{a=1}^k \frac{(u_{k+1}, v_a)}{(v_a, v_a)} v_a$$

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Gram-Schmidt is NOT numerically stable.