Factoring Matrices

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Overview

- Notation
- Our matrix and Different Factorizations of Matrices
- Example: Two Triangular Factorizations, LPU and LPL^T.
- More Triangular Factorizations explained
- Overview of Uses
  - Eigenvalues
  - Solving a system, eigenvectors
- My work

Notation

- L is lower triangular, often unit lower. Unit lower?
- U is upper triangular.
- P is a permutation matrix
- D is a diagonal matrix.
- L,D and U will sometimes be block matrices, clear from context.
- Ω will be an SSP matrix (signed symmetric permutation). More later.

Our matrix

- Start with a nonsingular, dense, n by n matrix A.

- doing $O(n^3)$ work so n should be no bigger than a few thousand.

- Our matrix A is too complicated. Get info from it by looking at its factors.

- A may also have singular pivots, minors, or otherwise ill behaved.

Factorizations

- Direct
  - $A = QR$ called QR factorization. This is used for linear system solving, least squares solutions, and finding eigenvalues.
  - $A = LU$ called LU factorization. This is used for linear system solving and finding eigenvalues.
  - $A = HR$ called HR and XHR factorization. This is used for finding eigenvalues.
- Iterative
  - $A = U\Sigma V'$ called Singular value decomposition.
  - $A = VDV^{-1}$ is the eigendecomposition.
  - $A = PR$ is the polar decomposition.

Triangular Factorizations and LU

An LU factorization of A is

$$\begin{aligned} A &= \begin{bmatrix} 1 & & & \\ * & \ddots & & \\ * & \ddots & \ddots & \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} * & * & * & * \\ & \ddots & * & * \\ & & \ddots & * \\ & & & * \end{bmatrix} \quad (1) \\ &= LU. \quad (2) \end{aligned}$$

Here L is unit lower triangular and U is upper triangular. All factorizations of A into triangular factors, with or without a permutation matrix involved are called $triangular\ factorizations$.

Triangular Factorizations and $L\Omega L^T$

A Generalized Choleski decomposition of a symmetric matrix A is

$$\begin{aligned} A &= \begin{bmatrix} * & & & \\ * & \ddots & & \\ * & \ddots & \ddots & \\ * & * & * & * \end{bmatrix} \Omega \begin{bmatrix} * & * & * & * \\ & \ddots & * & * \\ & & \ddots & * \\ & & & * \end{bmatrix} && (3) \\ &= L\Omega L^T. && (4) \end{aligned}$$

- Here L is lower triangular and Ω is an SSP matrix. SSP?
- Ω is basically a symmetric permutation matrix. So its got 0's and 1's and its symmetric. Sometimes it has -1's.

An example of $A = L\Omega L^T$

Let A be

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{5}$$

Find L_1 so that

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -l_{2,1} & 1 & 0 & 0 \\ -l_{3,1} & 0 & 1 & 0 \\ -l_{4,1} & 0 & 0 & 1 \end{bmatrix}, \tag{6}$$

and

$$L_1^{-1} A L_1^{-T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

An example of $A = L\Omega L^T$

Solving for these we find $l_{2,1} = l_{3,1} = 1$, and $l_{4,1} = 1/2$. The result $L_1^{-1}AL_1^{-T}$ has a non zero entry in the pivot position, $(2,2)$, so we look for L_2 so that

$$L_2^{-1}(L_1^{-1}AL_1^{-T})L_2^{-T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad (8)$$

where

$$L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -l_{3,2} & 1 & 0 \\ 0 & -l_{4,2} & 0 & 1 \end{bmatrix}. \qquad (9)$$

We find $l_{4,2} = 0$ and $l_{4,2} = 1$ leaving $* = -2$.

An example of $A = L\Omega L^T$

We can factor out the -2 and stick $\sqrt{2}$ in our triangular factors defined by $L = L_1 L_2 L_3$ and get

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

Next, we have

$$L^{-1} A L^{-T} = \Omega \tag{11}$$

$$L^{-1} A L^{-T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{12}$$

which implies $A = L\Omega L^T$.

An example of $A = L\Omega U$

Let A be the NONsymmetric matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{13}$$

Find L_1 and U_1 so that L_1 has the same structure as before and U_1^{-1} is

$$U_1^{-1} = \begin{bmatrix} 1 & -u_{1,2} & -u_{1,3} & -u_{1,4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

and

An example of $A = L\Omega U$

$$L_1^{-1} A U_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

Solving for these we find the same L_1 matrix as before, and $U_1 = L_1^T$. We find L_2 and U_2 so that

$$L_2^{-1}(L_1^{-1} A L_1^{-T}) L_2^{-T} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

We find $l_{4,2} = 0$, $l_{4,2} = 1$, $u_{2,3} = -1$, and $u_{2,4} = 0$, leaving $* = 2$. The difference here is that the $u_{2,3}$ is different from $l_{3,2}$ by a sign.

An example of $A = L\Omega U$

We end up with

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

and

$$U = \begin{bmatrix} 1 & 1 & 1 & 1/2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

The SSP matrix Ω is the same as before. $A = L\Omega U$. I $think$ that one can always do this. nonsingular A.

Various triangular factorizations

A nonsingular

- Modified Choleski: $A = L\Omega L^T$
- Bruhat decomposition: $A = LPU$
- Modified Bruhat decomposition: $A = L_1 P L_2$
- Made up from last example: $A = L\Omega U$(probably does not always exist)

A nonsingular, nonzero principle minors

- LU factorization $A = LU$.
- LDU and block LDU factorizations $A = LDU$.
- backwards UDL and UL factorizations $A = UL$ and $A = UDL$. The factorization $A = LDU$ is called a 'forward' triangular factorization in this context.

USES

Solving a system $Ax = b$ and finding the least squares solution

Solve the System:

- $A = LU$ where L is lower triangular and U is upper triangular.
- Finding the x so that $Ax = b$ means solving $Ly = b$ and then $Ux = y$ so that $LUx = Ly = b$.

Least Squares:

- $A = QR$ where Q is unitary$(QQ* = I)$ and R is upper triangular.
- Finding the x so that $Rx = Q * b$ means computing $Q * b$ and then solving.

QR and eigenvalues

- QR iteration. Its okay.

$$\begin{aligned} A_i &= Q_i R_i && (19) \\ A_{i+1} &= R_i Q_i && (20) \\ i &= i+1 && (21) \end{aligned}$$

- notice that $A_{i+1} = R_i Q_i = Q_i^T (Q_i R_i) Q_i = Q_i^T (A_i) Q_i$. A_i and A_{i+1} are similar so they have the same eigenvalues, A_{i+1} is more diagonal. Can add shifts to help convergence.

- $O(n^3)$ flops for one iteration? even if we just had one iteration per eigenvalue this is $O(n^4)$. reduce to Upper Hessenberg $(\frac{10}{3}n^3 + O(n^2))$ and then one iteration of QR is only $6n^2 + O(n)$ work.

- imagine LU factorization in its place

LR and eigenvalues

- LR iteration.

$$\begin{aligned} A_i &= L_i U_i && (22) \\ A_{i+1} &= U_i L_i && (23) \\ i &= i+1 && (24) \end{aligned}$$

- notice that $A_{i+1} = U_i L_i = L_i^{-1}(L_i U_i) L_i = L_i^{-1}(A_i) L_i$. This converges. Can add shifts to help convergence.

- imagine GR factorization in its place.

My work

- Two steps of LR equals one step of QR for SPD matrices. I showed a similar result.
- Solve for eigenvectors given forward and backward triangular factorizations of $A - \lambda I$.
- Investigate merit of a rewritten LR iteration called $dqds$ that contains no subtractions.
- Investigate eigenvalue algorithm that comes from any of the previously mentioned triangular factorizations.

Questions?

(or learn about twisted factorizations)

Twisted Factorizations

Let J be a tridiagonal matrix., then $J = L^+D^+U^+$ is a triangular factorization and

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ L^+_{1,1} & 1 & 0 & 0 \\ 0 & \ddots & 1 & 0 \\ 0 & 0 & L^+_{n,n-1} & 1 \end{bmatrix}, \tag{25}$$

and

$$U = \begin{bmatrix} 1 & U^+_{1,2} & 0 & 0 \\ 0 & 1 & \ddots & 0 \\ 0 & 0 & 1 & U^+_{n-1,1} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{26}$$

Suppose we also backward factor $J = U^-D^-L^-$. Then U^- and L^- have the same structure as U^+ and L^+, respectively. D^+ and D^- are diagonal matrices.

Twisted Factorizations

Suppose we have a matrix J in tridiagonal form and an approximate eigenvalue λ. Suppose further that we have the forward and backward triangular factorizations, of $J - \lambda I$, $U^- D^- L^- = J - \lambda I = L^+ D^+ U^+$. The matrices U^+ and U^- are unit upper triangular, L^- and L^+ are unit lower triangular, and D^+ and D^- are diagonal. A twisted factorization at the index k, of a tridiagonal matrix $J - \lambda I$ is defined as the factorization of $J - \lambda I = N_k D_k M_k$, where

$$N_k = \begin{bmatrix} 1 & & & & & & \\ L^+_{2,1} & \ddots & & & & & \\ & \ddots & 1 & & & & \\ & & L^+_{k,k-1} & 1 & U^-_{k,k+1} & & \\ & & & & 1 & \ddots & \\ & & & & & \ddots & U^-_{n-1,n} \\ & & & & & & 1 \end{bmatrix}, \qquad (27)$$

Twisted Factorizations

$$M_k = \begin{bmatrix} 1 & U^+_{1,2} & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & U^+_{k-1,k} & & & & \\ & & & 1 & & & & \\ & & & L^-_{k+1,k} & 1 & & & \\ & & & & \ddots & \ddots & & \\ & & & & & L^-_{n,n-1} & 1 \end{bmatrix} \tag{28}$$

$$D_k = \begin{bmatrix} D^+_{1,1} & & & & & & \\ & \ddots & & & & & \\ & & D^+_{k-1,k-1} & & & & \\ & & & \gamma_k & & & \\ & & & & D^-_{k+1,k+1} & & \\ & & & & & \ddots & \\ & & & & & & D^-_{n,n} \end{bmatrix}. \tag{29}$$

Here $\gamma_k = D^+(k) + D^-(k) - J(k,k) + \lambda$.

My work with these

- Calculate each γ_k. Find the smallest one.
- Use the twisted factorization at k. Solve $N_k D_k M_k z_k = e_k \gamma_k$ to find z_k.
- z_k is a very accurate eigenvector. No refinement is needed.
- $\frac{\gamma_k}{||z_k||}$ is the residuals and its often $O(\lambda_i - \lambda)$.