Bringing Eigenvalues to the People

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Motivation

- Dynamical systems: $\mathbf{x}' = f(\mathbf{x})$ stability of equilibrium solutions at $f(\mathbf{x}) = 0$ are based on evals of the jacobian$(\frac{\partial f_j}{\partial x_i})$.
- Stability of linear operators and their spectra [3].
- Time evolution:$Ax_n = x_{n+1}$ in math bio, Markov chains and probability, Fibonacci sequence.
- ODE's: convert $\sum_i p_i(x)y^{(i)} = 0$ into $A\mathbf{y} = \lambda\mathbf{y}'$ use discrete eigenvalues of A for asymptotics.
- Quadratic eigenvalue problem from structural mechanics: [12, 11]. This problem leads to nonsymmetric tridiagonals.

Overview

- Definitions
- Eigenvalue Problem
  - Symmetric
  - Nonsymmetric
- Backward error
- Forward error

Definitions

- λ is an eigenvalue and v is an eigenvector for a matrix A: $Av = \lambda v$
- A is a symmetric matrix: $A = A^T$
- U is a unitary matrix: $U^{-1} = U^*$.
- T is a tridiagonal matrix:

$$T = \begin{pmatrix} b_1 & a_1 & & & & \\ c_1 & b_2 & \ddots & & & \\ & \ddots & \ddots & a_i & & \\ & & c_i & b_{i+1} & \ddots & \\ & & & \ddots & \ddots & a_{n-1} \\ & & & & c_{n-1} & b_n \end{pmatrix} \tag{1}$$

Definitions

- A is an upper Hessenberg matrix:

$$A = \begin{pmatrix} b_1 & a_1 & * & * & * & * \\ c_1 & b_2 & \ddots & * & * & * \\ & \ddots & \ddots & a_i & * & * \\ & & c_i & b_{i+1} & \ddots & * \\ & & & \ddots & \ddots & a_{n-1} \\ & & & & c_{n-1} & b_n \end{pmatrix} \quad (2)$$

- $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$. A is positive definite if: $x^T A x > 0$ whenever $x \neq 0$.(Matrix Computations, Golub and Van Loan)

The General Eigenvalue Problem

- Symmetric case: Well mined
  - Divide and Conquer [6]
  - Bisection [4]
  - QR [8], [7], [4]
  - The Grail [5], [2]

- Nonsymmetric case: Not as well mined
  - Divide and Conquer [1]
  - QR
  - XHR [10], [9]

Typical Solution to the General Eigenvalue Problem

- Similarity reduction of A to tridiagonal matrix T(A symmetric) or upper Hessenberg H(A nonsymmetric).

$$\begin{aligned} A &= QTQ^T && (3) \\ A &= QHQ^T && (4) \end{aligned}$$

- Find eigenvectors and eigenvalues of the tridiagonal matrix T(upper Hessenberg matrix H if nonsymmetric) by reduction to diagonal matrix D or block upper triangular matrix R, by unitary matrices Q_i:

$$\begin{aligned} T &= Q_1Q_2...Q_kDQ_k^T...Q_2^TQ_1^T && (5) \\ H &= Q_1Q_2...Q_kRQ_k^T...Q_2^TQ_1^T && (6) \end{aligned}$$

- Back transform eigenvectors of T or H to get those of the matrix A.

Eigenvalues of a Symmetric Matrix A

- State problem: $Av = \lambda v$
- Absolute accuracy: $|\lambda_i(A + \delta A) - \lambda_i(A)| \le ||f(A)||$
- An example:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{7}$$

$\lambda(A) = -1, 3$ exactly. Let $\delta A = 1000 * \epsilon_{mach} * A$. Then we solve $A + \delta A$.

$$A + \delta A = \begin{pmatrix} 1 + 2.2 * 10^{-13} & 2 + 4.4 * 10^{-13} \\ 2 + 4.4 * 10^{-13} & 1 + 2 * 10^{-13} \end{pmatrix} \tag{8}$$

The larger eigenvalue $\lambda(A + \delta A) = 3(1 + 1000 * \epsilon_{mach})$, the smaller is $-1(1 + 1000 * \epsilon_{mach})$.

Eigenvalues of a Symmetric Matrix A

- In this case $f(A)$ was $||\delta A||$.

$$
\begin{aligned}
|\lambda_i(A) - \lambda_i(A + \delta A)| &= 3 - 3(1 + 1000 * \epsilon_{mach}) && (9)\\
&= 3000 * \epsilon_{mach} && (10)\\
&= ||\delta A|| && (11)
\end{aligned}
$$

- Weyl's Theorem states that $|\lambda_i(A) - \lambda_i(A + \delta A)| \leq ||\delta A||$ for symmetric A.

- Relative accuracy would be $\frac{|\lambda_i(A) - \lambda_i(A + \delta A)|}{|\lambda_i(A)|} \leq ||\delta A||$

- Relative accuracy is not always possible to achieve for small λ_i. But sometimes it is, even if the eval is 10^{-300}!.

Eigenvalues of a General Matrix A

- Accuracy results are not as good. If $A = SDS^{-1}$ with D diagonal and $A + E$ is a perturbation then(Bauer-Fike):

$$|\lambda(A) - \lambda(A + E)| \leq ||S|| \cdot ||S^{-1}|| \cdot ||E|| \quad (12)$$

- Condition of an eigenvalue is $\frac{1}{w^T v}$ where w is a left eigenvector, v is a right eigenvector: $w^T A = \lambda w^T$

- Lack of fast algorithms as well(unlike $O(n^2)$ algorithms for eigenvalues in the symmetric case). Due to upper Hessenberg form.(QR, orthogonal iteration)

Eigenvalues of a General Matrix A

Consider the following matrix:

$$A: \;=\; \begin{pmatrix} \mu & b \\ 0 & \lambda \end{pmatrix}$$

and the different perturbations

$$E_1 := \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix} \qquad E_2 := \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix} \qquad E_3 := \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$$

We have $\|E_j\|_\infty = \varepsilon$. The eigenvalues λ_j, μ_j of $A + E_j$ are

$$\lambda_1, \mu_1 \;=\; \frac{1}{2}\left(\mu + \lambda \pm \sqrt{(\mu - \lambda)^2 + 4b\varepsilon}\right)$$

$$\mu_2 = \mu + \varepsilon, \qquad \lambda_2 = \lambda$$

$$\mu_3 = \mu + \varepsilon, \qquad \lambda_3 = \lambda + \varepsilon$$

Backward error

- Backwards stability means that computed $\hat{L}$ and $\hat{U}$ solve a nearby problem$(L * U = A + \epsilon A)$

- theoretically LU has backward error:

$$\hat{L}\hat{U} = A + \delta A \quad (13)$$

$$\frac{||\delta A||}{||L||||U||} \leq O(e_{mach}) \quad (14)$$

- LU with pivoting has backward error:

$$\hat{L}\hat{U} = A + \delta A \quad (15)$$

$$\frac{||\delta A||}{||A||} \leq O(\rho e_{mach}) \quad (16)$$

where ρ is the growth factor that is bounded by 2^{n-1}.
(matlab:lubacktest2)

Forward error

- Forward error is the error between the exact solution L,U and the computed solution, $\hat{L}$, $\hat{U}$.
- go through the accumulation of errors and hope you get an answer.
- its hard to calculate because it depends on the condition of the problem, usually a global thing.
- what about Gaussian Elimination?
- matlab:lufortest

Bibliography

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