THE RESULT OF TWO STEPS OF THE LR ALGORITHM IS DIAGONALLY SIMILAR TO THE RESULT OF ONE STEP OF THE HR ALGORITHM

JASON SLEMONS*

Abstract. Real nonsymmetric tridiagonal matrices arise in various applications. When one is asked to find the eigenvalues of such a matrix, the QR algorithm is used but this destroys tridiagonal form by converting the matrix to Hessenberg form, resulting in increased storage requirements and numerical operations. The HR algorithm, based on the HR factorization of the matrix into a (Δ, Δ_1)-orthogonal part H, where $H^T\Delta H = \Delta_1$, and an upper triangular part R, solves this problem. In a result proved by Hongguo Xu, two steps of the LR algorithm are equivalent to one step of the QR algorithm for symmetric matrices. The first object of this note is to use the HR algorithm to extend Hongguo Xu's result to the nonsymmetric case. Since an HR factorization does not always exist so we also consider an extension to it called XHR factorization. We then prove a similar result about it.

Key words. LR, QR, triangular factorization, HR

AMS subject classifications. 15A18 15A23 15A57 65F15

1. Introduction.

An attractive homework exercise in a course on Matrix Computations is to show that for an SPD (Symmetric Positive Definite) matrix A, the result of one step of the QR algorithm is equal to the result of two steps of the Cholesky LR algorithm. In [7] Hongguo Xu proves that if a symmetric matrix A admits a triangular factorization, then result of one step of the QR algorithm applied to A is equivalent to the result of two steps of the LR algorithm applied to A. The goal of this note is to extend Hongguo Xu's result to the nonsymmetric case. This goal seems doomed because the QR algorithm does not preserve bandwidth while the LR algorithm does. The way out of this difficulty is to find another algorithm that is less restrictive than QR. This useful step was taken in 1981 by Angelika Bunse-Gerstner in [3] with the introduction of the HR algorithm.

2. *HR*.

The HR factorization of a matrix requires the use of marices that when squared yeild the identity. In this paper we consider two such classes of matrices; signature matrices, the set of which is denoted by Δ, and signed symmetric permutation matrices.

Definition 2.1. *We define Ω to be an* ***SSP*** *(signed symmetric permutation) matrix if it is symmetric and, for a symmetric permutation π,*

$$\Omega_{i,j} = \begin{cases} \pm 1 & if\ j = \pi(i) \\ 0 & otherwise \end{cases} \tag{2.1}$$

For example,

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{2.2}$$

is an SSP matrix. Clearly signature matrices are also SSP matrices.

*Supported at the University of Washington Applied Mathematics Department by the National Science Foundation grant number DMS 0354131.

In this paper Δ and Ω will denote a signature matrix and an SSP matrix, respectively.

DEFINITION 2.2. *Let $H \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and let Ω_0, Ω be SSP matrices. We say that H is (Ω, Ω_0)-**orthogonal** if $H^T\Omega H = \Omega_0$, and denote by $O(\Omega, \Omega_0)$ the set of all (Ω, Ω_0)-orthogonal matrices. When Δ and Δ_0 are signature matrices we define (Δ, Δ_0)-**orthogonal** in the same way. Furthermore $O(\Delta, \Delta_0)$ is the set of all H such that $H^T\Delta H = \Delta_0$.*

Observe that if H is an element of $O(\Omega, \Omega_0)$ then $H^{-1} = \Omega_0 H^T \Omega$. This will be useful in Theorem 2.5.

DEFINITION 2.3. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. A is Ω-**symmetric** if $A^T\Omega = \Omega A$, and Δ- **symmetric** if $A^T\Delta = \Delta A$.*

DEFINITION 2.4. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular $\Delta-$symmetric matrix. The **HR factorization** of A is $A = HR$ with $H \in O(\Delta, \Delta_0)$, where $R \in \mathbb{R}^{n\times n}$ is an upper triangular matrix that has positive diagonal entries.*

The HR factorization exists if and only if no principle minor of $A^T\Delta A$ vanishes and the product of the first i diagonal entries of Δ_0 coincides with the sign of the i^{th} principle minor of $A^T\Delta A$ for each $i \in \{1...n\}$. The HR factorization is also unique. Analyses of its use in the HR algorithm can be found in [1], [2], and [3].

2.1. *HR* and *LR*.

Given the factorization of a matrix $A = LU$ where L is a unit lower triangular matrix and U is an upper triangular matrix, we define the result of one step of the LR algorithm applied to A to be the matrix given by UL. This matrix is similar to the original by $UL = L^{-1}AL$. The result of one step of the HR algorithm, on a matrix A given its HR factorization, $A = HR$, is the matrix RH which is similar to A by $RH = H^{-1}AH$. Throughout this section we will denote the matrix UL as $\dot{A}$. The result of two steps of the LR algorithm applied to A means finding $\dot{A}$, factoring it as $\dot{A} = \hat{L}\hat{U}$, and then forming $\ddot{A} = \hat{U}\hat{L}$. $\ddot{A}$ is similar to $\dot{A}$ which is similar to A, and in turn $\ddot{A}$ is similar to A. As we continue the algorithm, at each step the result has the same eigenvalues as the original matrix.

The real use of the LR or HR algorithms is in thier application to tridiagonal matrices. If T_G is an unreduced tridiagonal matrix, we can assume without loss of generality that it is balanced[1]. Next we consider $T_G = \Delta T$, its factorization into a signature matrix, Δ, and a symmetric tridiagonal matrix T. This can be done by simply factoring out ± 1 from each row. T_G is then Δ-symmetric, and assuming it has the required factorizations, we can apply the following theorem to it.

THEOREM 2.5. *Let T_G be a real nonsingular matrix and Δ a signature matrix. Furthermore assume T_G and $\dot{T}_G$ have triangular factorizations. If T_G is Δ-symmetric then one step of the HR algorithm applied to T_G is similar by a diagonal matrix to $\ddot{T}_G$.*

Proof. If T_G is Δ-symmetric then $T_G^T\Delta = \Delta T_G$. $T = \Delta T_G$ is a symmetric matrix. Since $T_G = \Delta T$, the existense of a triangular factorization of T_G implies the existense of a triangular factorization for T. Let $T = L_1 D_1 L_1^T$, where L_1 is a unit lower triangular matrix and D_1 is a diagonal matrix. We factorize T_G as

$$\begin{aligned} T_G &= \Delta T \\ &= (\Delta L_1 \Delta)(\Delta D_1 L_1^T), \end{aligned}$$

[1]This means T_G's ith absolute row sum equals its ith absolute column sum. Every tridiagonal matrix is similar by a diagonal matrix to a balanced tridiagonal matrix, meaning $|T_{G_{i,i+1}}| = |T_{G_{i+1,i}}|$.

which is the LR factorization of T_G. The result of one step of the LR algorithm applied to T_G is

$$\dot{T}_G = (\Delta D_1 L_1^T)(\Delta L_1 \Delta). \tag{2.3}$$

Since $\dot{T}_G$ permits triangular factorization then so does $L_1^T \Delta L_1$. We decompose $L_1^T \Delta L_1 = L_2 D_2 L_2^T$. Given $L_1^T \Delta L_1 = L_2 D_2 L_2^T$ we can now explicitly seperate the triangular factors of $\dot{T}_G$ to get

$$\begin{aligned} \dot{T}_G &= \Delta D_1 L_1^T \Delta L_1 \Delta \\ &= \Delta D_1 L_2 D_2 L_2^T \Delta \\ &= (\Delta D_1 L_2 D_1^{-1} \Delta)(\Delta D_1 D_2 L_2^T \Delta). \end{aligned} \tag{2.4}$$

The matrix $\Delta D_1 L_2 D_1^{-1} \Delta$ is unit lower triangular and $\Delta D_1 D_2 L_2^T \Delta$ is an upper triangular matrix so (2.4) is the LR factorization of $\dot{T}_G$. Applying another step of the LR algorithm to $\dot{T}_G$ we get

$$\ddot{T}_G = (\Delta D_1 D_2 L_2^T \Delta)(\Delta D_1 L_2 D_1^{-1} \Delta). \tag{2.5}$$

The matrix $\ddot{T}_G$ denotes the result of two steps of the LR algorithm applied to T_G. It remains to show that (2.5) is diagonally similar to one step of the HR algorithm applied to T_G. For this purpose, we rewrite $D_2 = \bar{D}_2 \Delta_2 \bar{D}_2$, where $\bar{D}_2 > 0$ is a diagonal matrix, and Δ_2 is a signature matrix. This allows us to define the matrix Q as $L_1 L_2^{-T} \bar{D}_2^{-1}$. Notice that

$$\begin{aligned} \bar{D}_2^{-1} L_2^{-1} L_1^T \Delta L_1 L_2^{-T} \bar{D}_2^{-1} &= \Delta_2 \\ Q^T \Delta Q &= \Delta_2, \end{aligned} \tag{2.6}$$

implies Q is (Δ, Δ_2)-orthogonal. Another factorization of Q, coming from (2.6) is

$$\begin{aligned} Q &= \Delta Q^{-T} \Delta_2 \\ &= \Delta L_1^{-T} L_2 \bar{D}_2 \Delta_2. \end{aligned} \tag{2.7}$$

Next, we gradually manipulate (2.5) to introduce Q and Q^{-1}. First replace the diagonal matrix $\Delta D_1 \bar{D}_2 \Delta_2$ with $\mathcal{D}$. Then from (2.5) and (2.7) we have

$$\begin{aligned} \ddot{T}_G &= \mathcal{D} \bar{D}_2 L_2^T D_1 L_2 \bar{D}_2 \Delta_2 \mathcal{D}^{-1} \\ &= \mathcal{D}(\bar{D}_2 L_2^T L_1^{-1})(L_1 D_1 L_2 \bar{D}_2 \Delta_2)\mathcal{D}^{-1} \\ &= \mathcal{D} Q^{-1} L_1 D_1 (L_1^T L_1^{-T}) L_2 \bar{D}_2 \Delta_2 \mathcal{D}^{-1} \\ &= \mathcal{D} Q^{-1} (L_1 D_1 L_1^T)(L_1^{-T} L_2 \bar{D}_2 \Delta_2)\mathcal{D}^{-1} \\ &= \mathcal{D} Q^{-1} T \Delta Q \mathcal{D}^{-1} && (2.8) \\ &= \mathcal{D} (\Delta Q)^{-1} T_G (\Delta Q) \mathcal{D}^{-1}. && (2.9) \end{aligned}$$

Since $\Delta^3 = \Delta$, it follows that $(\Delta Q)^T \Delta (\Delta Q) = \Delta_2$, so ΔQ is also (Δ, Δ_2)-orthogonal. Finally we manipulate $T_G = \Delta T$ into its HR factorization,

$$\begin{aligned} T_G &= \Delta L_1 D_1 L_1^T \\ &= \Delta L_1 (L_2^{-T} \bar{D}_2^{-1})(\bar{D}_2 L_2^T) D_1 L_1^T \\ &= \Delta Q (\bar{D}_2 L_2^T D_1 L_1^T). \end{aligned}$$

(2.10)

Write D_1 as $|D_1|sign(D_1)$ to find

$$T_G = \Delta Q sign(D_1)[sign(D_1)\bar{D_2}L_2^T sign(D_1)]|D_1|L_1^T. \tag{2.11}$$

Notice that $H = \Delta Q sign(D_1)$ is (Δ, Δ_2)-orthogonal and $R = sign(D_1)\bar{D_2}L_2^T sign(D_1)]|D_1|L_1^T$ is an upper triangular matrix with positive diagonal elements. Therefore equation (2.11) exhibits the HR factorization of T_G. Using (2.9) we discover

$$\ddot{T}_G = sign(D_1)\mathcal{D}H^{-1}T_G H(sign(D_1)\mathcal{D})^{-1}. \tag{2.12}$$

The result of one step of the HR algorithm applied to T_G is $H^{-1}T_G H = RH$. Note that (2.12) exhibits the diagonal similarity mentioned in the statement of the theorem between $\ddot{T}_G$ and this matrix. □

In an effort to extend this result to multiple steps of the HR algorithm we define $sign(D_1)\mathcal{D} = \mathcal{D}_1$, $H = H_1$, and $X = \mathcal{D}_1^{-1}\ddot{T}_G\mathcal{D}_1 = H_1^{-1}T_G H_1$, using the same notation as in the proof of Theorem 2.5. Since $H_1 \in O(\Delta, \Delta_2)$ and $H_1^{-1}T_G H_1 = \Delta_2 H_1^T T H_1$, X is Δ_2-symmetric. Therefore we can apply the HR algorithm to X and get $H_2^{-1}XH_2$, where $H_2 \in O(\Delta_2, \Delta_3)$. By Theorem 2.5,

$$\ddot{X} = \mathcal{D}_2 H_2^{-1} X H_2 \mathcal{D}_2^{-1},$$

where $\ddot{X}$ is the result of two steps of the LR algorithm applied to X, and $\mathcal{D}_2$ is a diagonal matrix. It is a straightforward exercise to show that

$$\ddot{X} = \mathcal{D}_1^{-1}\ddot{\ddot{T}}_G\mathcal{D}_1,$$

where $\ddot{\ddot{T}}_G$ is the result of four steps of the LR algorithm applied to T_G. This result then leads to

$$\begin{aligned}\ddot{\ddot{T}}_G &= \mathcal{D}_1\mathcal{D}_2 H_2^{-1} X H_2 \mathcal{D}_2^{-1}\mathcal{D}_1^{-1}\\ &= \mathcal{D}_1\mathcal{D}_2 H_2^{-1} H_1^{-1} T_G H_1 H_2 \mathcal{D}_2^{-1}\mathcal{D}_1^{-1}.\end{aligned}$$

Since $H_2 \in O(\Delta_2, \Delta_3)$ and $H_1 \in O(\Delta, \Delta_2)$ then $H_1^{-1} = \Delta_2 H_1^T \Delta$ and $H_2^{-1} = \Delta_3 H_2^T \Delta_2$. Therefore

$$H_2^{-1}H_1^{-1}T_G H_1 H_2 = \Delta_3(H_1H_2)^T T(H_1H_2).$$

The result of two steps of the HR algorithm, $H_2^{-1}H_1^{-1}T_G H_1 H_2$, is Δ_3 symmetric and so we could apply the theorem yet again. One can prove that the result, on a nonsingular Δ-symmetric matrix, of $2k$ steps of the LR algorithm is diagonally similar to the result of k steps of the HR algorithm on the same matrix, provided that at each stage the factorizations exist.

2.2. *XHR* and *SSPLR*. Unfortunately the HR factorization does not always exist for a matrix A. The condition that H be an element of $O(\Delta, \Delta_0)$ is too restrictive. Fortunately if the space is expanded to $H \in O(\Omega, \Omega_0)$ such a factorization always exists; see [6] and [4].

THEOREM 2.6. *Let A be nonsingular, and let Ω be an SSP matrix. The factorization $A = HR$ always exists for some $H \in O(\Omega, \Omega_0)$, where Ω_0 is an SSP matrix,* *and some $R \in \mathbb{R}^{n\times n}$, an upper triangular matrix with positive diagonal entries. Such a factorization is called the* ***XHR factorization*** *of A with respect to Ω. Moreover Ω_0 is uniquely determined by A and Ω.*

Proof. See Liu, [4]. □

Is there an algorithm based on triangular factorization other than the LR algorithm that is related to to a step of the XHR algorithm and not the HR algorithm? Since every nonsingular matrix has an XHR factorization, the algorithm and the triangular factorization it comes from will be more flexible than the LR factorization. For this reason we relax the conditions on our triangular factors. The following theorem is very much like the modified Bruhat Decomposition in [5], but differs in that it is specific to SSP matrices.

THEOREM 2.7. *If A is a real nonsingular symmetric matrix, then A can be decomposed as $A = L\Omega L^T$. Here L is unit lower triangular and $\Omega = D\bar{\Omega}D$ where D is a positive diagonal matrix and $\bar{\Omega}$ is an SSP matrix. In addition, $\bar{\Omega}$ is unique. We will call this an SSPLR factorization.*

Proof. This is slightly modified from the factorization in [4] but the proof is the same. See Liu [4]. □

We define the $SSPLR$ factorization of a general matrix A as a factorization of A into $L\Omega U$, but this may not always exits. In this context L is a unit lower triangular matrix , U is an upper triangular matrix and Ω is an SSP matrix. The result of a single step of the $SSPLR$ algorithm applied to A is the matrix $\dot{A} = \Omega UL$. Notice that $\dot{A} = LAL^{-1}$, so the eigenvalues of $\dot{A}$ and A are the same. The $SSPLR$ factorization is a triangular factorization that is more flexible than the LR factorization since it always exists for nonsingular symmetric matrices. The XHR factorization is a similar generalization of HR. We define a single step of the XHR algorithm applied to a matrix A given its XHR factorization $A = HR$ to be the matrix RH.

Now the question of whether or not a single step of the XHR algorithm is equivalent to a combination of steps of the $SSPLR$ and LR algorithms can be answered.

THEOREM 2.8. *Let T_G be a real nonsingular matrix with an LR factorization, and let Δ be a signature matrix. If T_G is Δ-symmetric then the result of one step of the XHR algorithm applied to T_G is similar, by an SSP matrix times a diagonal matrix, to the result of one step of the LR algorithm applied to A followed by one step of the SSPLR algorithm.*

Proof. As before, we can write either an unreduced tridiagonal matrix or a Δ-symmetric matrix T_G as ΔT with T symmetric. Since T_G has an LR factorization we may factor T into unit triangular matrices L_1 and L_1^T, and a diagonal matrix D_1 as $L_1 D_1 L_1^T$. Consider one step of the LR algorithm applied to $T_G = \Delta T$. We get

$$\begin{aligned} T_G &= \Delta T \\ &= \Delta L_1 \Delta \Delta D_1 L_1^T \\ \dot{T}_G &= \Delta D_1 L_1^T \Delta L_1 \Delta. \end{aligned} \tag{2.13}$$

The $SSPLR$ factorization of the symmetric matrix $L_1^T \Delta L_1$ is $L_2 \Omega_2 L_2^T$. The matrix L_2 is unit lower triangular and $\Omega_2 = \bar{D}_2 \bar{\Omega}_2 \bar{D}_2$ where $\bar{D}_2$ is a positive diagonal matrix, and $\bar{\Omega}_2$ is an SSP matrix. This factorization always exists by Theorem 2.7. We take another step to get

$$\begin{aligned} \dot{T}_G &= \Delta D_1 L_2 \Omega_2 L_2^T \Delta \\ &= (\Delta D_1 L_2 D_1^{-1} \Delta)(\Delta D_1 \Omega_2 L_2^T \Delta) && (2.14) \\ \ddot{T}_G &= (\Delta D_1 \Omega_2 L_2^T \Delta)(\Delta D_1 L_2 D_1^{-1} \Delta). && (2.15) \end{aligned}$$

Notice that $\Delta D_1 L_2 D_1^{-1}\Delta$ is a unit lower triangular matrix and $\Delta D_1 \Omega_2 L_2^T \Delta$ can be written as an SSP matrix times an upper triangular matrix. Thus, equation (2.14) exhibits the $SSPLR$ factorization of the matrix $\dot{T}_G$. $\ddot{T}_G$ is the result of one step of the $SSPLR$ algorithm applied to $\dot{T}_G$. It remains to show that $\ddot{T}_G$ is similar to the result of one step of the XHR algorithm, as opposed to one step of the HR algorithm. We manipulate the triangular factorization of T_G by inserting the identity to get its XHR factorization,

$$T_G = \Delta L_1 (L_2^{-T} \bar{D}_2^{-1} sign(D_1))(sign(D_1)\bar{D}_2 L_2^T) D_1 L_1^T, \tag{2.16}$$

where $sign(D_1)|D_1| = D_1$. Observe that $R = sign(D_1)\bar{D}_2 L_2^T D_1 L_1^T$ is an upper triangular matrix with positive diagonal elements. The rest of the factorization above is H since

$$\begin{aligned}(\Delta L_1 L_2^{-T} \bar{D}_2^{-1} sign(D_1))^T \Delta (\Delta L_1 L_2^{-T} \bar{D}_2^{-1} sign(D_1)) &= \\ sign(D_1) Q^T \Delta\Delta\Delta Q sign(D_1) &= \\ sign(D_1) Q^T \Delta Q sign(D_1) &= \\ sign(D_1)\bar{\Omega}_2 sign(D_1) &= \Omega_3.\end{aligned}$$

We use the definition of Q from the proof of Theorem 2.5. If $H = \Delta L_1 L_2^{-T} \bar{D}_2^{-1} sign(D_1)$ then the above calculation implies H is (Δ, Ω_3)-orthogonal. This fact along with (2.16) implies that H and R are the XHR factors of T_G. Since Ω_3 is an SSP matrix and not a signature matrix this cannot be an HR factorization. It remains to show that $H^{-1} T_G H$ is similar, by an SSP matrix times a diagonal matrix, to $\ddot{T}_G$. Evaluating (2.15) we get

$$\begin{aligned}\ddot{T}_G &= (\Delta D_1 \Omega_2 L_2^T \Delta)(\Delta D_1 L_2 D_1^{-1} \Delta) \\ &= \Delta D_1 \bar{D}_2 \bar{\Omega}_2 \bar{D}_2 L_2^T D_1 L_2 D_1^{-1} \Delta\end{aligned}$$

since $\Omega_2 = \bar{D}_2 \bar{\Omega}_2 \bar{D}_2$. Inserting $sign(D_1)^2$ and $L_1^{-1}\Delta^2 L_1$ we get

$$\begin{aligned}\ddot{T}_G &= \Delta D_1 \bar{D}_2 \bar{\Omega}_2 sign(D_1)(sign(D_1)\bar{D}_2 L_2^T L_1^{-1} \Delta)\Delta L_1 D_1 L_2 D_1^{-1} \Delta \\ &= \Delta D_1 \bar{D}_2 \bar{\Omega}_2 sign(D_1) H^{-1} \Delta L_1 D_1 L_2 D_1^{-1} \Delta\end{aligned}$$

because $H^{-1} = sign(D_1)\bar{D}_2 L_2^T L_1^{-1}\Delta$. Define $\mathcal{D} = sign(D_1)\bar{\Omega}_2 \bar{D}_2^{-1} D_1^{-1} \Delta$, and $\mathcal{D}^{-1} = \Delta D_1 \bar{D}_2 \bar{\Omega}_2 sign(D_1)$ to get

$$\begin{aligned}\ddot{T}_G &= \mathcal{D}^{-1} H^{-1} \Delta L_1 D_1 L_2 \bar{D}_2 \bar{\Omega}_2 sign(D_1) \mathcal{D} \\ &= \mathcal{D}^{-1} H^{-1} (\Delta L_1 D_1 L_1^T)(L_1^{-T} L_2 \bar{D}_2 \bar{\Omega}_2 sign(D_1))\mathcal{D}.\end{aligned}$$

As a result of $H^T \Delta H = \Omega_3$ we also know that $H = \Delta H^{-T} \Omega_3$ This implies

$$\begin{aligned}\ddot{T}_G &= \mathcal{D}^{-1} H^{-1} (\Delta L_1 D_1 L_1^T) H \mathcal{D}. \\ &= \mathcal{D}^{-1} H^{-1} T_G H \mathcal{D}.\end{aligned} \tag{2.17}$$

The similarity matrix is $\mathcal{D}$, which can be written as an SSP matrix times a diagonal matrix. □

Unfortunately $H^{-1} T_G H = \Omega_3 H^T T H$ is an Ω_3-symmetric matrix but Ω_3 is not a signature matrix and so we cannot apply this theorem again in the way we did for Theorem 2.5.

The author would like to thank Beresford Parlett and Loyce Adams for their helpful comments on the manuscript. The author would also like to thank the referees for their helpful suggestions.

REFERENCES

[1] P. Benner, H. Faßbender, and D.S. Watkins, *Two connections between the SR and HR eigenvalue algorithms*, Linear Algebra and Its Applications, 272 (1998), pp. 17–32.

[2] M.A. Brebner and J. Grad, *Eigenvalues of $Ax{=}B\lambda\, x$ for real symmetric matrices A and B by reduction to pseudosymmetric form and the HR process*, Linear Algebra and Its Applications, 43 (1982), pp. 99–118.

[3] A. Bunse-Gerstner, *An analysis of the HR algorithm for computing the eigenvalues of a matrix*, Linear Algebra and Its Applications, 35 (1981), pp. 155–173.

[4] Zhi-Shun Liu, *On the extended HR algorithm*, Tech. Report 564, Center for Pure and Applied Mathematics, U.C. Berkeley, 1992.

[5] E. Tyrtyshnikov, *Matrix Bruhat decompositions with a remark on the QR(GR) algorithm*, Linear Algebra and Its Applications, 250 (1997), pp. 61–68.

[6] P. Wiberg, *A study of the HR and extended HR methods for the standard eigenvalue problem*, Tech. Report TR/PA/97/33, CERFACS, Toulouse, France, 1997.

[7] Hongguo Xu, *The relation between the QR and LR algorithms*, SIAM J. Matrix Anal. Appl., 19 (1998), pp. 551–555.